Appendix A

Some continued fraction expansions

This is a catalogue of some of the known continued fraction expansions. The list is in no way complete. Still it can be useful, both to find a continued fraction expansion of some given function and to "sum" a given continued fraction.

We have not attempted to find the origin of each result. The references we give are therefore just pointing to books or papers where the expansion also can be found.

A.1 Introduction

A.1.1 Notation

We write

$$f(\cdots) = \mathbf{K}(a_n(\cdots)/b_n(\cdots)); \quad (\cdots) \in D \tag{1.1.1}$$

to say that the continued fraction converges in the classical sense to $f(\cdots)$ for the parameters $(\cdots)$ in the set D. In the literature the set D is often far too restrictive, if it is given at all. We have determined a (possibly larger) set D_c where the continued fraction converges. This is done by methods presented in this book. However, it may well happen that the equality (1.1.1) fails in a subset of D_c, even if $f(\cdots)$ is interpreted as an analytic continuation of the expression in question. In some cases we therefore give expressions for sets D_c and D_f such that $\mathbf{K}(a_n(\cdots)/b_n(\cdots))$ converges in D_c and the equality holds in $D_f \subseteq D_c$. What happens outside these sets is not checked. The identities normally holds also at points where $a_n(\cdots) = 0$ unless otherwise stated.

The elements $a_n(\cdots)$ and $b_n(\cdots)$ of almost all continued fractions in this appendix are polynomials in the parameters. The classical approximants are then rational functions of the parameters, and can therefore not converge to multivalued functions. If the left hand side of (1.1.1) is a multivalued function, we always take the principal part unless otherwise

L. Lorentzen and H. Waadeland, *Continued Fractions*, Atlantis Studies in Mathematics for Engineering and Science 1, DOI 10.2991/978-94-91216-37-4,

stated. The principal part is often written with a capital first letter, such as Ln z, Arctan z etc.

A.1.2 Transformations

It is evident that not every continued fraction expansion can find room in a book like this. On the other hand, quite a number of the known continued fraction expansions can be derived from one another by simple transformations. We have for instance

$$f = b_0 + \mathbf{K}(a_n/b_n) \Longleftrightarrow c \cdot \frac{1}{f} = \frac{c}{b_0} \, \frac{}{+} \, \frac{a_1}{b_1} \, \frac{}{+} \, \frac{a_2}{b_2} \, \frac{}{+} \cdots ; \quad c \neq 0. \tag{1.2.1}$$

Similarly, if $f = b_0 + \mathbf{K}(a_n/b_n)$, then $g = (f-1)/(f+1) = 1 - 2/(1+f)$; i.e.,

$$f = b_0 + \mathbf{K}(a_n/b_n) \Longleftrightarrow \frac{f-1}{f+1} = 1 - \frac{2}{1+b_0} \, \frac{}{+} \, \frac{a_1}{b_1} \, \frac{}{+} \, \frac{a_2}{b_2} \, \frac{}{+} \cdots . \tag{1.2.2}$$

Another simple transformation is maybe most easily described for S-fractions. Assume that $f(z) = b_0 + \mathbf{K}(a_n z/1)$. Then $f(z^{-1}) = b_0 + \mathbf{K}(a_n z^{-1}/1)$. Equivalence transformations lead to

$$\begin{aligned} f\left(\tfrac{1}{z}\right) &= b_0 + \frac{a_1}{z} \, \frac{}{+} \, \frac{a_2}{1} \, \frac{}{+} \, \frac{a_3}{z} \, \frac{}{+} \, \frac{a_4}{1} \, \frac{}{+} \, \frac{a_5}{z} \, \frac{}{+} \cdots \\ &= b_0 + \frac{a_1/z}{1} \, \frac{}{+} \, \frac{a_2}{z} \, \frac{}{+} \, \frac{a_3}{1} \, \frac{}{+} \, \frac{a_4}{z} \, \frac{}{+} \, \frac{a_5}{1} \, \frac{}{+} \cdots \\ &= b_0 + \frac{a_1/\xi}{\xi} \, \frac{}{+} \, \frac{a_2}{\xi} \, \frac{}{+} \, \frac{a_3}{\xi} \, \frac{}{+} \, \frac{a_4}{\xi} \, \frac{}{+} \, \frac{a_5}{\xi} \, \frac{}{+} \cdots , \end{aligned}$$

where $\xi^2 = z$. We shall normally not list equivalent continued fractions like this separately.

Another situation that often arises is the following: We have

$$f(z) = b_0 + \frac{a_1 z^2}{1} \, \frac{}{+} \, \frac{a_2 z^2}{1} \, \frac{}{+} \, \frac{a_3 z^2}{1} \, \frac{}{+} \cdots . \tag{1.2.3a}$$

Then

$$f(iz) = b_0 - \frac{a_1 z^2}{1} \, \frac{}{-} \, \frac{a_2 z^2}{1} \, \frac{}{-} \, \frac{a_3 z^2}{1} \, \frac{}{-} \cdots . \tag{1.2.4b}$$

Of course, every time we have a continued fraction expansion $f = b_0 + \mathbf{K}(a_n/b_n)$ with all $a_n, b_n \neq 0$, we can take its even or odd part and obtain a "new" continued fraction converging to the same value f. Some of these variations will be listed, in particular if they turn out to be nice and simple.

A.2 Elementary functions

A.2.1 Mathematical constants

$$\pi = 3 + \frac{1}{7} \, \frac{}{+} \, \frac{1}{15} \, \frac{}{+} \, \frac{1}{1} \, \frac{}{+} \, \frac{1}{292} \, \frac{}{+} \, \frac{1}{1} \, \frac{}{+} \, \frac{1}{1} \, \frac{}{+} \, \frac{1}{1} \, \frac{}{+} \, \frac{1}{2} \, \frac{}{+} \, \frac{1}{1} \, \frac{}{+} \, \frac{1}{3} \, \frac{}{+} \, \frac{1}{1} \, \frac{}{+} \, \frac{1}{14} \, \frac{}{+} \cdots , \tag{2.1.1}$$

([JoTh80], p 23). This is the regular continued fraction expansion of π.

$$\pi = \frac{4}{1+}\,\frac{1^2}{3+}\,\frac{2^2}{5+}\,\frac{3^2}{7+}\,\frac{4^2}{9+\cdots}, \tag{2.1.2}$$

([JoTh80], p 25), (see also (3.6.1)).

$$\frac{\pi}{2} = 1 + \frac{1}{1+}\,\frac{1\cdot 2}{1+}\,\frac{2\cdot 3}{1+}\,\frac{3\cdot 4}{1+\cdots}, \quad \text{([Khru06a])}. \tag{2.1.3}$$

For the **Riemann zeta function** we have

$$\tfrac{1}{2}\zeta(2) = \frac{\pi^2}{12} = \frac{1}{1+}\,\frac{1^4}{3+}\,\frac{2^4}{5+}\,\frac{3^4}{7+\cdots}, \tag{2.1.4}$$

([Bern89], p 150).

$$\zeta(2) = \frac{\pi^2}{6} = 1 + \frac{1}{1+}\,\frac{1^2}{1+}\,\frac{1\cdot 2}{1+}\,\frac{2^2}{1+}\,\frac{2\cdot 3}{1+}\,\frac{3^2}{1+}\,\frac{3\cdot 4}{1+}\,\frac{4^2}{1+\cdots}, \tag{2.1.5}$$

([Bern89], p 153).

Apery's constant:

$$\zeta(3) = 1 + \frac{1}{4+}\,\frac{1^3}{1+}\,\frac{1^3}{12+}\,\frac{2^3}{1+}\,\frac{2^3}{20+}\,\frac{3^3}{1+}\,\frac{3^3}{28+}\,\frac{4^3}{1+}\,\frac{4^3}{36+\cdots}, \tag{2.1.6}$$

([Bern89], p 155), (see also (4.7.37)).

Euler's number:

$$e = \frac{1}{1-}\,\frac{1}{1+}\,\frac{1}{2-}\,\frac{1}{3+}\,\frac{1}{2-}\,\frac{1}{5+}\,\frac{1}{2-}\,\frac{1}{7+\cdots}, \tag{2.1.7}$$

([JoTh80], p 25), (see also (3.2.1)).

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+\cdots}, \tag{2.1.8}$$

([JoTh80], p 23).

$$e = 1 + \frac{2}{1+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\,\frac{1}{18+\cdots}, \tag{2.1.9}$$

([Khov63], p 114). (See also (3.2.2)).

$$e = 2 + \frac{2}{2+}\,\frac{3}{3+}\,\frac{4}{4+}\,\frac{5}{5+\cdots}, \tag{2.1.10}$$

([Perr57], p 57).

$$e = \frac{1}{1-}\,\frac{2}{3+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\,\frac{1}{18+\cdots}, \tag{2.1.11}$$

([Khov63], p 114). (See also (3.2.2) for $z = -1$).

$$\sqrt{e} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{9+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{13+\cdots}, \tag{2.1.12}$$

([Euler37]).

$$\sqrt[3]{e} = 1 + \frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{8+}\frac{1}{1+}\frac{1}{1+}\frac{1}{14+}\frac{1}{1+}\frac{1}{1+}\frac{1}{20+}\cdots, \tag{2.1.13}$$

([Euler37]). (See also (2.2.4).)

$$\coth\tfrac{1}{2} = \frac{e+1}{e-1} = 2 + \frac{1}{6+}\frac{1}{10+}\frac{1}{14+}\frac{1}{18+}\cdots, \tag{2.1.14}$$

([Khru06b]).

The golden ratio:

$$\frac{\sqrt{5}-1}{2} = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots, \tag{2.1.15}$$

([JoTh80], p 23).

Catalan's constant: $G := \sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$:

$$2G = 2 - \frac{1^2}{3+}\frac{2^2}{1+}\frac{2^2}{3+}\frac{4^2}{1+}\frac{4^2}{3+}\frac{6^2}{1+}\frac{6^2}{3+}\cdots, \tag{2.1.16}$$

([Bern89], p 151). (See also (4.7.30) with $z := 2$.)

$$2G = 1 + \frac{1}{1/2+}\frac{1^2}{1/2+}\frac{1\cdot 2}{1/2+}\frac{2^2}{1/2+}\frac{2\cdot 3}{1/2+}\frac{3^2}{1/2+}\frac{3\cdot 4}{1/2+}\cdots, \tag{2.1.17}$$

([Bern89], p 153). (See also (4.7.32) with $z := \frac{1}{2}$.)

A.2.2 The exponential function

$$\begin{aligned} e^z &= {}_1F_1(1;1;z) = \frac{1}{1-}\frac{z}{1+}\frac{z}{2-}\frac{z}{3+}\frac{z}{2-}\frac{z}{5+}\frac{z}{2-}\frac{z}{7+}\cdots \\ &= \frac{1}{1-}\frac{z}{1+}\frac{1z}{2-}\frac{1z}{3+}\frac{2z}{4-}\frac{2z}{5+}\frac{3z}{6-}\frac{3z}{7+}\frac{4z}{8-}\frac{4z}{9+}\cdots; \quad z \in \mathbb{C}, \end{aligned}$$

([JoTh80], p 207). (See also (4.1.4).) The odd part of this continued fraction is

$$e^z = 1 + \frac{2z}{2-z+}\frac{z^2}{6+}\frac{z^2}{10+}\frac{z^2}{14+}\frac{z^2}{18+}\cdots; \quad z \in \mathbb{C}, \tag{2.2.1}$$

([Khov63], p 114).

$$e^z = 1 + \frac{z}{1-z+}\frac{1z}{2-z+}\frac{2z}{3-z+}\frac{3z}{4-z+}\cdots; \; z \in \mathbb{C}, \tag{2.2.2}$$

([JoTh80], p 272).

Since $e^z = 1/e^{-z}$, we can find three more expansions from (2.2.1)–(2.2.2) by use of (1.2.1). For instance, (2.2.2) transforms into

$$e^z = \frac{1}{1-}\frac{z}{1+z-}\frac{1z}{2+z-}\frac{2z}{3+z-}\frac{3z}{4+z+}\cdots; \; z \in \mathbb{C}, \tag{2.2.3}$$

([Khov63], p 113).

$$e^{1/z} = 1 + \frac{1}{z-1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3z-1+}\frac{1}{1+}\cdots\frac{1}{1+}\frac{1}{5z-1+}\frac{1}{1+}\cdots \quad z \in \mathbb{C}, \tag{2.2.4}$$

([Khru06b]).

Lambert's continued fraction

$$\frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{z}{1+}\frac{z^2}{3\ +}\frac{z^2}{5\ +}\frac{z^2}{7\ +}\cdots\ ; z \in \mathbb{C}, \tag{2.2.5}$$

([Wall48], p 349), is easily obtained from (2.2.1) by use of (1.2.2).

A.2.3 The general binomial function

The general binomial function $(1+z)^\alpha$ is a multivalued function. As already mentioned in the introduction, we shall always let $(1+z)^\alpha$ mean the principal part of this function; i.e., as always,

$$(1+z)^\alpha := \exp(\alpha \operatorname{Ln}(1+z)) \text{ where } -\pi < \operatorname{Im}(\operatorname{Ln}(1+z)) \le \pi.$$

We then have the following expansions:

$$\begin{aligned}(1+z)^\alpha &= {}_2F_1(-\alpha, 1; 1; -z)\\ &= \frac{1}{1-}\frac{\alpha z}{1\ +}\frac{(1+\alpha)z}{2\quad +}\frac{(1-\alpha)z}{3\quad +}\frac{(2+\alpha)z}{2\quad +}\frac{(2-\alpha)z}{5\quad +}\frac{(3+\alpha)z}{2\quad +}\frac{(3-\alpha)z}{7\quad +}\cdots\end{aligned} \tag{2.3.1}$$

for $\alpha \in \mathbb{C}$ and $|\arg(z+1)| < \pi$, ([JoTh80], p 202). (See also (3.1.6).) The odd part of this continued fraction is

$$\begin{aligned}&(1+z)^\alpha =\\ &1 + \frac{2\alpha z}{2+(1-\alpha)z-}\frac{(1^2-\alpha^2)z^2}{3(z+2)\quad -}\frac{(2^2-\alpha^2)z^2}{5(z+2)\quad -}\frac{(3^2-\alpha^2)z^2}{7(z+2)\quad -}\cdots\end{aligned} \tag{2.3.2}$$

for $\alpha \in \mathbb{C}$ and $|\arg(z+1)| < \pi$, ([Khov63], p 105). (2.3.2) is also the odd part of

$$\begin{aligned}&(1+z)^\alpha =\\ &\frac{1}{1-}\frac{\alpha z}{1(1+z)-}\frac{(1-\alpha)z}{2\quad -}\frac{(1+\alpha)z}{3(1+z)-}\frac{(2-\alpha)z}{2\quad -}\frac{(2+\alpha)z}{5(1+z)-}\frac{(3-\alpha)z}{2\quad -}\cdots\end{aligned} \tag{2.3.3}$$

for $\alpha \in \mathbb{C}$ and $|\arg(z+1)| < \pi$, ([Khov63], p 101).

$$(1+z)^\alpha = \frac{1}{1-}\frac{\alpha z}{1+(1+\alpha)z-}\frac{1(1+\alpha)z(1+z)}{2+(3+\alpha)z\quad -}\frac{2(2+\alpha)z(1+z)}{3+(5+\alpha)z\quad -}\cdots \tag{2.3.4}$$

([Khov63], p 101). $D_c := \{(\alpha, z) \in \mathbb{C}^2; \operatorname{Re}(z) \neq -\frac{1}{2},\ z \neq -1\}$, $D_f := \{(\alpha, z) \in \mathbb{C}^2; \operatorname{Re}(z) > -\frac{1}{2}\}$.

$$(1+z)^\alpha = \frac{1}{1-}\frac{\alpha z}{1+\alpha z+}\frac{1(1-\alpha)z}{2-(1-\alpha)z+}\frac{2(2-\alpha)z}{3-(2-\alpha)z+}\frac{3(3-\alpha)z}{4-(3-\alpha)z+}\cdots, \tag{2.3.5}$$

([Khov63], p 102). $D_c := \{(\alpha, z) \in \mathbb{C}^2;\ |z| \neq 1\}$, $D_c := \{(\alpha, z) \in \mathbb{C}^2;\ |z| < 1\}$.

The general binomial function satisfies $(1+z)^\alpha = 1/(1+z)^{-\alpha}$. Hence the equality (1.2.1) applied to these 5 expansions gives us 5 new ones. To find a continued fraction expansion for

$$\left(\frac{z+1}{z-1}\right)^\alpha = \left(1 + \frac{2}{z-1}\right)^\alpha \tag{2.3.6}$$

we can use any of the 5 expansions (2.3.1)–(2.3.5) with z replaced by $2/(z-1)$.

Laguerre's continued fraction

$$\left(\frac{z+1}{z-1}\right)^\alpha = 1 + \frac{2\alpha}{z-\alpha+}\,\frac{\alpha^2-1^2}{3z}\,\underset{+}{}\,\frac{\alpha^2-2^2}{5z}\,\underset{+}{}\,\frac{\alpha^2-3^2}{7z}\,\underset{+}{}\cdots, \tag{2.3.7}$$

for $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus [-1, 1]$, ([Perr57], p 153).

$$\begin{aligned} z^{1/z} = 1 + \frac{z-1}{1z} &\underset{+}{} \frac{(1z-1)(z-1)}{2} \underset{+}{} \frac{(1z+1)(z-1)}{3z} \underset{+}{} \\ \frac{(2z-1)(z-1)}{2} &\underset{+}{} \frac{(2z+1)(z-1)}{5z} \underset{+}{} \frac{(3z-1)(z-1)}{2} \underset{+}{} \frac{(3z+1)(z-1)}{7z} \underset{+}{}\cdots \end{aligned} \tag{2.3.8}$$

for $|\arg z| < \pi$, ([Khov63], p 109). The even part of (2.3.8) is

$$\begin{aligned} z^{1/z} = 1 + &\frac{2(z-1)}{z^2+1} \underset{-}{} \frac{(1^2z^2-1)(z-1)^2}{3z(z+1)} \underset{-}{} \\ &\frac{(2^2z^2-1)(z-1)^2}{5z(z+1)} \underset{-}{} \frac{(3^2z^2-1)(z-1)^2}{7z(z+1)} \underset{-}{}\cdots \end{aligned} \tag{2.3.9}$$

for $|\arg z| < \pi$, ([Khov63], p 110).

$$\begin{aligned} \left(\frac{1+az}{1+bz}\right)^\alpha = 1 + &\frac{2\alpha(a-b)z}{2+(a+b-\alpha(a-b))z -} \, \frac{(a-b)^2(1^2-\alpha^2)z^2}{3(2+(a+b)z)} \underset{-}{} \\ &\frac{(a-b)^2(2^2-\alpha^2)z^2}{5(2+(a+b)z)} \underset{-}{} \frac{(a-b)^2(3^2-\alpha^2)z^2}{7(2+(a+b)z)} \underset{-}{}\cdots \end{aligned} \tag{2.3.10}$$

for $\alpha \in \mathbb{C}$ and $\frac{2+(a+b)z}{(a-b)z} \in \mathbb{C} \setminus [-1, 1]$, ([Perr57], p 264). (Note that $\frac{2+(a+b)z}{(a-b)z} = \frac{x+y}{x-y}$ for $x := 1 + az$, $y := 1 + bz$.)

A.2.4 The natural logarithm

$$\begin{aligned} \mathrm{Ln}(1+z) &= z\,{}_2F_1(1, 1; 2; -z) = z \int_0^1 \frac{dt}{1+zt} \\ &= \frac{z}{1+}\,\frac{1z}{2+}\,\frac{1z}{3+}\,\frac{2z}{2+}\,\frac{2z}{5+}\,\frac{3z}{2+}\,\frac{3z}{7+}\,\frac{4z}{2+}\,\frac{4z}{9+}\cdots \\ &= \frac{z}{1+}\,\frac{1^2z}{2+}\,\frac{1^2z}{3+}\,\frac{2^2z}{4+}\,\frac{2^2z}{5+}\,\frac{3^2z}{6+}\,\frac{3^2z}{7+}\,\frac{4^2z}{8+}\,\frac{4^2z}{9+}\cdots \end{aligned} \tag{2.4.1}$$

for $|\arg(1+z)| < \pi$, ([JoTh80], p 203). (See also (3.1.6).)

$$\mathrm{Ln}(1+z) = \frac{z}{1+z} \, \frac{z}{-1} \, \frac{1}{+1+z} \, \frac{z}{-1} \, \frac{1/2}{+1+z} \, \frac{z}{-1} \, \frac{1}{+1+z} \, \frac{z}{-1} \, \frac{2/3}{+1+z} - \cdots \tag{2.4.2}$$

for $|\arg(1+z)| < \pi$, ([JoTh80], p 319). Here the continued fraction has the form $\mathbf{K}(a_n(z)/b_n(z))$ where all $a_{2n}(z) = -z$, $a_{4n-1}(z) = 1$ and $a_{4n+1}(z) = n/(n+1)$. (2.4.1) is the even part of (2.4.2). The odd part of (2.4.2) can be written

$$\begin{aligned}\mathrm{Ln}(1+z) &= \frac{z}{1+z}\left\{1 + \frac{z}{2} \, \frac{2z}{+3} \, \frac{z}{+2} \, \frac{3z}{+5} \, \frac{2z}{+2} \, \frac{4z}{+7} \, \frac{3z}{+2} \, \frac{5z}{+9} \, \frac{4z}{+2} + \cdots\right\} \\ &= \frac{z}{1+z}\left\{1 + \frac{z}{2} \, \frac{1\cdot 2z}{+3} \, \frac{1\cdot 2z}{+4} \, \frac{2\cdot 3z}{+5} \, \frac{2\cdot 3z}{+6} \, \frac{3\cdot 4z}{+7} \, \frac{3\cdot 4z}{+8} + \cdots\right\}\end{aligned} \tag{2.4.3}$$

for $|\arg(1+z)| < \pi$. The even part of (2.4.1) is

$$\mathrm{Ln}(1+z) = \frac{2z}{1(2+z)} \, \frac{1^2z^2}{-3(2+z)} \, \frac{2^2z^2}{-5(2+z)} \, \frac{3^2z^2}{-7(2+z)} - \cdots \tag{2.4.4}$$

for $|\arg(1+z)| < \pi$, ([Khov63], p 111).

$$\mathrm{Ln}(1+z) = \frac{z}{1} \, \frac{1^2z}{+2-z} \, \frac{2^2z}{+3-2z} \, \frac{3^2z}{+4-3z} \, \frac{4^2z}{+5-4z} \, \frac{5^2z}{+6-5z} + \cdots, \tag{2.4.5}$$

([Khov63], p 111). $D_c := \{z \in \mathbb{C};\ |z| \neq 1\}$, $D_f := \{z \in \mathbb{C};\ |z| < 1\}$.

The connection

$$\mathrm{Ln}(1+z) = -\mathrm{Ln}\left(\frac{1}{1+z}\right) = -\mathrm{Ln}\left(1 - \frac{z}{1+z}\right) \tag{2.4.6}$$

can be applied to (2.4.1)–(2.4.5) to get 5 new continued fraction expansions. For instance, from (2.4.1) we get

$$\mathrm{Ln}(1+z) = \frac{z}{1+z} \, \frac{1z}{-2} \, \frac{1z}{-3(1+z)} \, \frac{2z}{-2} \, \frac{2z}{-5(1+z)} \, \frac{3z}{-2} \, \frac{3z}{-7(1+z)} - \cdots \tag{2.4.7}$$

for $|\arg(1+z)| < \pi$, ([Khov63], p 110), and from (2.4.5)

$$\mathrm{Ln}(1+z) = \frac{z}{1+z} \, \frac{1^2z(1+z)}{-2+3z} \, \frac{2^2z(1+z)}{-3+5z} \, \frac{3^2z(1+z)}{-4+7z} - \cdots, \tag{2.4.8}$$

([Khov63], p 111). $D_c := \{z \in \mathbb{C};\ \mathrm{Re}(z) \neq -\frac{1}{2}\}$, $D_f := \{z \in \mathbb{C};\ \mathrm{Re}(z) > -\frac{1}{2}\}$.

$$\begin{aligned}\mathrm{Ln}\Big(\frac{1+z}{1-z}\Big) &= 2z\,{}_2F_1(\tfrac{1}{2},1;\tfrac{3}{2};z^2) = \mathrm{Ln}\Big(1+\frac{2z}{1-z}\Big) = z\int_{-1}^{1}\frac{dt}{1+tz} \\ &= \frac{2z}{1} \, \frac{1^2z^2}{-3} \, \frac{2^2z^2}{-5} \, \frac{3^2z^2}{-7} \, \frac{4^2z^2}{-9} - \cdots\end{aligned} \tag{2.4.9}$$

for $|\arg(1-z^2)| < \pi$, ([JoTh80], p 203). (See also (3.1.6).) From this we also get

$$\mathrm{Ln}\Big(\frac{z+1}{z-1}\Big) = \mathrm{Ln}\Big(\frac{1+1/z}{1-1/z}\Big) = \frac{2}{z} \, \frac{1^2}{-3z} \, \frac{2^2}{-5z} \, \frac{3^2}{-7z} \, \frac{4^2}{-9z} - \cdots \tag{2.4.10}$$

for $z \in \mathbb{C} \setminus [-1,1]$, ([Perr57], p 155). Of course, also other continued fraction expansions for $\mathrm{Ln}(1+z)$ can be used to derive expressions for $\mathrm{Ln}((1+z)/(1-z))$.

A.2.5 Trigonometric and hyperbolic functions

$$\tan z := \frac{\sin z}{\cos z} = z\frac{{}_0F_1(3/2;-z^2/4)}{{}_0F_1(1/2;-z^2/4)} = \frac{z}{1-}\,\frac{z^2}{3-}\,\frac{z^2}{5-}\,\frac{z^2}{7-}\,\frac{z^2}{9-\cdots}\,;\ z\in\mathbb{C} \tag{2.5.1}$$

([JoTh80], p 211), (See also (3.1.1).) The odd part of (2.5.1) is

$$\begin{aligned}\tan z = z + \frac{5z^3}{1\cdot 3\cdot 5 - 6z^2 -}\,\frac{1\cdot 9z^4}{5\cdot 7\cdot 9 - 14z^2 -}\\ \frac{5\cdot 13z^4}{9\cdot 11\cdot 13 - 22z^2 -}\,\frac{9\cdot 17z^4}{13\cdot 15\cdot 17 - 30z^2 -\cdots}\,;\ \ z\in\mathbb{C},\end{aligned} \tag{2.5.2}$$

Another type of expansion is

$$\tan\frac{z\pi}{4} = \frac{z}{1+}\,\frac{1^2-z^2}{2\ +}\,\frac{3^2-z^2}{2\ +}\,\frac{5^2-z^2}{2\ +}\,\frac{7^2-z^2}{2\ +\cdots}\,;\ \ z\in\mathbb{C}, \tag{2.5.3}$$

([Perr57], p 35). From these expansions one also gets continued fractions for $\cot z = 1/\tan z$, $\tanh z = -i\tan(iz)$ and $\coth z = i/\tan(iz)$.

Quite another type of expansion for $\tan z$ follows from the identity

$$\tan\alpha z = -i\frac{(1+i\tan z)^\alpha - (1-i\tan z)^\alpha}{(1+i\tan z)^\alpha + (1-i\tan z)^\alpha} = -i\,\frac{y-1}{y+1}, \tag{2.5.4}$$

where $y := ((1+i\tan z)/(1-i\tan z))^\alpha$ can be expanded according to (2.3.7). Combined with (1.2.2) we get

$$\tan\alpha z = \frac{\alpha\tan z}{1\ \ -}\,\frac{(\alpha^2-1^2)\tan^2 z}{3\ \ -}\,\frac{(\alpha^2-2^2)\tan^2 z}{5\ \ -}\,\frac{(\alpha^2-3^2)\tan^2 z}{7\ \ -\cdots}\,, \tag{2.5.5}$$

([Khov63], p 108). $D_c := \{(\alpha,z)\in\mathbb{C}^2; \mathrm{Re}(\cos z)\neq 0\}$, $D_f := \{(\alpha,z)\in\mathbb{C}^2; |\mathrm{Re}(z)| < \pi/2\}$.

$$\coth\tfrac{1}{z} = 1 + \frac{1}{3z+}\,\frac{1}{5z+}\,\frac{1}{7z+}\,\frac{1}{9z+\cdots}\,; z\in\mathbb{C}, \tag{2.5.6}$$

([Khru06b]).

$$\frac{\pi z}{2}\coth\frac{\pi z}{2} = 1 + \frac{z^2}{1\ +}\,\frac{1^2(z^2+1^2)}{3\ \ +}\,\frac{2^2(z^2+2^2)}{5\ \ +}\,\frac{3^2(z^2+3^2)}{7\ \ +\cdots} \tag{2.5.7}$$

for all $z\in\mathbb{C}$, ([ABJL92], entry 44).

$$\frac{a\tanh(\pi b/2) - b\tanh(\pi a/2)}{a\tanh(\pi a/2) - b\tanh(\pi b/2)} = \frac{ab}{1\ +}\,\frac{(a^2+1^2)(b^2+1^2)}{3\ \ +}\,\frac{(a^2+2^2)(b^2+2^2)}{5\ \ +\cdots} \tag{2.5.8}$$

for all $a,b\in\mathbb{C}$, ([ABJL92], entry 47).

$$\frac{\sinh(\pi z)-\sin(\pi z)}{\sinh(\pi z)+\sin(\pi z)} = \frac{2z^2}{1\ +}\,\frac{4z^4+1^4}{3\ \ +}\,\frac{4z^4+2^4}{5\ \ +}\,\frac{4z^4+3^4}{7\ \ +\cdots} \tag{2.5.9}$$

for all $z\in\mathbb{C}$, ([ABJL92], entry 49).

A.2.6 Inverse trigonometric and hyperbolic functions

$$\begin{aligned}\operatorname{Arctan} z &= z\,{}_2F_1(\tfrac{1}{2},1;\tfrac{3}{2};-z^2) = -\frac{i}{2}\operatorname{Ln}\Big(\frac{1+iz}{1-iz}\Big)\\ &= \frac{z}{1+}\,\frac{1^2z^2}{3}\,{+}\,\frac{2^2z^2}{5}\,{+}\,\frac{3^2z^2}{7}\,{+}\,\frac{4^2z^2}{9}\,{+}\cdots\end{aligned} \tag{2.6.1}$$

for $|\arg(1+z^2)|<\pi$; i.e. $z\in\mathbb{C}\setminus i((-\infty,-1]\cup[1,\infty))$, ([JoTh80], p 202). (See also (3.1.6).) This continued fraction can also be written

$$\operatorname{Arctan} z = z - \frac{z^3}{3+}\,\frac{3^2z^2}{5}\,{+}\,\frac{2^2z^2}{7}\,{+}\,\frac{5^2z^2}{9}\,{+}\,\frac{4^2z^2}{11}\,{+}\,\frac{7^2z^2}{13}\,{+}\,\frac{6^2z^2}{15}\,{+}\cdots \tag{2.6.2}$$

for $z\in\mathbb{C}\setminus i((-\infty,-1]\cup[1,\infty))$, ([Khov63], p 117).

$$\operatorname{Arctan} z = \frac{z}{1(1+z^2)-}\,\frac{1\cdot 2z^2}{3}\,{-}\,\frac{1\cdot 2z^2}{5(1+z^2)-}\,\frac{3\cdot 4z^2}{7}\,{-}\,\frac{3\cdot 4z^2}{9(1+z^2)-}\,\frac{5\cdot 6z^2}{11}\,{-}\cdots \tag{2.6.3}$$

for $z\in\mathbb{C}\setminus i((-\infty,-1]\cup[1,\infty))$, ([Khov63], p 121). (This follows from (2.6.6) with z replaced by $z(1+z^2)^{-1/2}$.) Since $\operatorname{Artanh} z = i\operatorname{Arctan}(-iz)$, we also get continued fraction expansions for $\operatorname{Artanh} z$ from (2.6.1)– (2.6.3).

Also expressions for

$$\operatorname{Arcsin} z = \operatorname{Arctan}\Big(\frac{z}{\sqrt{1-z^2}}\Big), \quad \operatorname{Arccos} z = \operatorname{Arctan}\Big(\frac{\sqrt{1-z^2}}{z}\Big)$$

can be obtained. For instance, from (2.6.1) we get

$$\frac{\operatorname{Arcsin} z}{\sqrt{1-z^2}} = \frac{z}{1(1-z^2)+}\,\frac{1^2z^2}{3}\,{+}\,\frac{2^2z^2}{5(1-z^2)+}\,\frac{3^2z^2}{7}\,{+}\,\frac{4^2z^2}{9(1-z^2)+}\cdots \tag{2.6.4}$$

for $|\arg(1-z^2)|<\pi$, ([Khov63], p 118) and

$$\frac{\operatorname{Arccos} z}{\sqrt{1-z^2}} = \frac{1}{z+}\,\frac{1^2(1-z^2)}{3z}\,{+}\,\frac{2^2(1-z^2)}{5z}\,{+}\,\frac{3^2(1-z^2)}{7z}\,{+}\cdots, \tag{2.6.5}$$

([Khov63], p 119). Here $D_c := \{z\in\mathbb{C};\operatorname{Re} z\neq 0\}$ and $D_f := \{z\in\mathbb{C};\operatorname{Re} z>0\}$. We also have

$$\begin{aligned}\frac{\operatorname{Arcsin} z}{\sqrt{1-z^2}} &= z\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2)}{{}_2F_1(\frac{1}{2},-\frac{1}{2};\frac{1}{2};z^2)}\\ &= \frac{z}{1-}\,\frac{1\cdot 2z^2}{3}\,{-}\,\frac{1\cdot 2z^2}{5}\,{-}\,\frac{3\cdot 4z^2}{7}\,{-}\,\frac{3\cdot 4z^2}{9}\,{-}\,\frac{5\cdot 6z^2}{11}\,{-}\,\frac{5\cdot 6z^2}{13}\,{-}\cdots\end{aligned} \tag{2.6.6}$$

for $|\arg(1-z^2)|<\pi$, ([JoTh80], p 203), and thus, since $\operatorname{Arccos} z = \operatorname{Arcsin}\sqrt{1-z^2}$ for $0\le z\le\frac{\pi}{2}$

$$\frac{\text{Arccos z}}{\sqrt{1-z^2}} = \frac{z}{1-}\,\frac{1\cdot 2(1-z^2)}{3}\,{-}\,\frac{1\cdot 2(1-z^2)}{5}\,{-}\,\frac{3\cdot 4(1-z^2)}{7}\,{-}\,\frac{3\cdot 4(1-z^2)}{9}\,{-}\cdots, \tag{2.6.7}$$

([Khov63], p 121) where $D_c := \{z \in \mathbb{C};\ \mathrm{Re}(z) \neq 0\}$, $D_f := \{z \in \mathbb{C};\ \mathrm{Re}(z) > 0\}$.

Similar expressions for inverse hyperbolic functions can be derived, since $\mathrm{Arsinh}\, z = i\mathrm{Arcsin}(-iz)$ and $(\mathrm{Arcosh}\, z)/\sqrt{z^2-1} = (\mathrm{Arccos}\, z)/\sqrt{1-z^2}$ for $0 \le z \le \frac{\pi}{2}$.

A neat formula can be obtained from (3.2.6) in the following way

$$\begin{aligned}\left(\frac{iz+1}{iz-1}\right)^{i\alpha} &= \exp\left(i\alpha \mathrm{Ln}\left(\frac{iz+1}{iz-1}\right)\right) = \exp(2\alpha \mathrm{Arctan}(1/z)) \\ &= 1 + \frac{2\alpha}{z-\alpha+}\,\frac{\alpha^2+1^2}{3z}\,\frac{}{+}\,\frac{\alpha^2+2^2}{5z}\,\frac{}{+}\,\frac{\alpha^2+3^2}{7z}\,\frac{}{+\cdots}\end{aligned} \tag{2.6.8}$$

for $|\arg(1+1/z^2)| < \pi$, i.e. $z \notin i[-1,1]$, ([Wall57], p 346).

$$\frac{\mathrm{Arsinh}\, z}{(1+z^2)^{1/2}} = z\,{}_2F_1(1,1;\tfrac{3}{2};-z^2) = \frac{z}{1+}\,\frac{2z^2}{1\ +}\,\frac{2(1+z^2)}{1\ \ +}\,\frac{4z^2}{1\ +}\,\frac{4(1+z^2)}{1\ \ +\cdots} \tag{2.6.9}$$

for $z \in \mathbb{R}$, ([ABJL92], entry 37).

$$\mathrm{Arctan}\, z = z\,{}_2F_1(\tfrac{1}{2},1;\tfrac{3}{2};-z^2) = \frac{z}{1+}\,\frac{1z^2}{1\ +}\,\frac{2(1+z^2)}{1\ \ +}\,\frac{3z^2}{1\ +}\,\frac{4(1+z^2)}{1\ \ +\cdots} \tag{2.6.10}$$

for $z \in \mathbb{R}$, ([ABJL92], entry 38).

A.2.7 Continued fractions with simple values

The continued fractions in this section all have easy to find tail sequences ([Lore08c]). The first one is taken from ([Perr57], p 279). It converges for all pairs $(a,z) \in \mathbb{C}^2$, but the equality

$$0 = -a - z + \frac{z}{1-a-z+}\,\frac{2z}{2-a-z+}\,\frac{3z}{3-a-z+\cdots} \tag{2.7.1}$$

given by Perron is only claimed for $z \neq 0$ and $a \in \mathbb{N} \cup \{0\}$. In our next continued fraction the constant sequence $\{1\}$ is always a tail sequence, so

$$1 = \frac{z+1}{z\ \ +}\,\frac{z+2}{z+1+}\,\frac{z+3}{z+2+}\,\frac{z+4}{z+3+\cdots}; \quad z \in \mathbb{C} \setminus (-\mathbb{N}), \tag{2.7.2}$$

([Bern89], p 112). (See also (3.1.5) with $z := 1$, $a := z+1$ and $c := z+1$.) The sequence of tail values for the next continued fraction is also known ([Lore08c]); it is in fact $t_0 := 1$, $t_{n+1} := z + na$ for $n \ge 0$. Hence

$$1 = \frac{z+a}{a\ \ +}\,\frac{(z+a)^2-a^2}{a\ \ \ +}\,\frac{(z+2a)^2-a^2}{a\ \ \ +}\,\frac{(z+3a)^2-a^2}{a\ \ \ +\cdots} \tag{2.7.3}$$

for $a \neq 0$ and $z/a \notin (-\mathbb{N})$, ([Bern89], p 118). (See also (3.1.8) with $z := -1$, $a := 0$, $b := (z/a) - 2$ and $c := z/a$.)

$$a = \frac{ab}{a+b+d-}\,\frac{(a+d)(b+d)}{a+b+3d}\,-\,\frac{(a+2d)(b+2d)}{a+b+5d}\,-\cdots, \tag{2.7.4}$$

([Bern89], p 119). Here $D_c := \{(a,b,d) \in \mathbb{C}^3;\ \mathrm{Re}((a-b)/d) \neq 0 \text{ or } a = b\}$. Both $\{-a-nd\}$ and $\{-b-nd\}$ are tail sequences, so the value of the continued fraction is either a or b. The value is a in $D_f := \{(a,b,d) \in \mathbb{C}^3;\ \mathrm{Re}((a-b)/d) < 0 \text{ or } a = b\}$. (See also (3.1.6) with $z = 1$, a replaced by $(a+d)/2d$, b replaced by $a/2d$ and c replaced by $(a+b+d)/2d$.) For $b = a$ replaced by $a+1$ and $d := 1$, (3.7.4) can be transformed into

$$a = 2a+1 - \frac{(a+1)^2}{2a+3}\,-\,\frac{(a+2)^2}{2a+5}\,-\,\frac{(a+3)^2}{2a+7}\,-\cdots; \quad a \in \mathbb{C}, \tag{2.7.5}$$

([Perr57], p 105).

$$az = \frac{abz}{b-(a+1)z+}\,\frac{(a+1)(b+1)z}{b+1-(a+2)z+}\,\frac{(a+2)(b+2)z}{b+2-(a+3)z+}\cdots, \tag{2.7.6}$$

([Perr57], p 290). $D_c := \{(a,b,z) \in \mathbb{C}^3;\ |z| \neq 1 \text{ and } b \neq 0,-1,-2,\dots\}$, $D_f := \{(a,b,z) \in D_c;\ |z| < 1\}$. (See also (3.1.8) with z replaced by $-z$, a replaced by $b-a$, and $b = c$ replaced by $b-1$.)

$$\frac{z+a+1}{z+1} = \frac{z+a}{z-1+}\,\frac{z+2a}{z+a-1+}\,\frac{z+3a}{z+2a-1+}\cdots, \tag{2.7.7}$$

([Bern89], p 115). $D_c := \{(a,z) \in \mathbb{C}^2;\ a \neq 0 \text{ and } z/a \neq 0,-1,-2,\dots\} \cup \{(a,z) \in \mathbb{C}^2;\ a = 0 \text{ and } |z| \neq 1\}$, $D_f := \{(a,z) \in D_c;\ \text{if } a = 0, \text{ then } |z| > 1\}$. (See also (3.1.5) with z replaced by $1/a$, a replaced by $z/a+1$ and c replaced by z/a.) If we instead let $z = 1$ and replace a by $z-1$, c by $z-3$ in (3.1.5) we get

$$\frac{z^2+z+1}{z^2-z+1} = \frac{z}{z-3+}\,\frac{z+1}{z-2+}\,\frac{z+2}{z-1+}\,\frac{z+3}{z}\,+\,\frac{z+4}{z+1+}\cdots, \tag{2.7.8}$$

([Bern89], p 118). $D_c := \mathbb{C}$, $D_f := \{z \in \mathbb{C};\ z \neq 0,-1,-2,\dots\}$. For $z := 1, a := z-1$ and $c := z-4$ in (3.1.5) we get

$$\frac{z^3+2z+1}{(z-1)^3+2(z-1)+1} = \frac{z}{z-4+}\,\frac{z+1}{z-3+}\,\frac{z+2}{z-2+}\,\frac{z+3}{z-1+}\,\frac{z+4}{z}\,+\cdots, \tag{2.7.9}$$

([Bern89], p 118). $D_c := \mathbb{C}$, $D_f := \{z \in \mathbb{C};\ z \neq 0,-1,-2,\dots\}$.

A.3 Hypergeometric functions

A.3.1 General expressions

$$c\frac{{}_0F_1(c;z)}{{}_0F_1(c+1;z)} = c + \frac{z}{c+1+}\,\frac{z}{c+2+}\,\frac{z}{c+3+}\cdots; \quad (c,z) \in \mathbb{C}^2, \tag{3.1.1}$$

([JoTh80], p 210).

$$\frac{{}_2F_0(a,b;z)}{{}_2F_0(a,b+1;z)} = 1 - \frac{az}{1} \underset{-}{} \frac{(b+1)z}{1} \underset{-}{} \frac{(a+1)z}{1} \underset{-}{} \frac{(b+2)z}{1} \underset{-}{} \frac{(a+2)z}{1} \underset{-}{} \cdots \tag{3.1.2}$$

for $(a,b,z) \in \mathbb{C}^3$ with $|\arg(-z)| < \pi$, ([JoTh80], p 213). The even part of this one is

$$\frac{{}_2F_0(a,b;z)}{{}_2F_0(a,b+1;z)} = 1 + \frac{az}{(b+1)z-1} \underset{-}{} \frac{(a+1)(b+1)z^2}{(a+b+3)z-1} \underset{-}{} \frac{(a+2)(b+2)z^2}{(a+b+5)z-1} \underset{-}{} \cdots \tag{3.1.3}$$

for $(a,b,z) \in \mathbb{C}^3$ with $|\arg(-z)| < \pi$.

$$\begin{aligned} c \cdot \frac{{}_1F_1(a;c;z)}{{}_1F_1(a+1;c+1;z)} &= c - \frac{(c-a)z}{c+1} \underset{+}{} \frac{(a+1)z}{c+2} \\ &\underset{-}{} \frac{(c-a+1)z}{c+3} \underset{+}{} \frac{(a+2)z}{c+4} \underset{-}{} \frac{(c-a+2)z}{c+5} \underset{+}{} \cdots; \quad (a,c,z) \in \mathbb{C}^3, \end{aligned} \tag{3.1.4}$$

([JoTh80], p 206).

$$\frac{{}_1F_1(a+1;c+1;z)}{{}_1F_1(a;c;z)} = \frac{c}{c-z} \underset{+}{} \frac{(a+1)z}{c+1-z} \underset{+}{} \frac{(a+2)z}{c+2-z} \underset{+}{} \frac{(a+3)z}{c+3-z} \underset{+}{} \cdots \tag{3.1.5}$$

for $(a,c,z) \in \mathbb{C}^3$, ([JoTh80], p 278).

$$\begin{aligned} c \cdot \frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a,b+1;c+1;z)} &= c - \frac{a(c-b)z}{c+1} \underset{-}{} \frac{(b+1)(c-a+1)z}{c+2} \\ &\underset{-}{} \frac{(a+1)(c-b+1)z}{c+3} \underset{-}{} \frac{(b+2)(c-a+2)z}{c+4} \underset{-}{} \frac{(a+2)(c-b+2)z}{c+5} \underset{-}{} \cdots \end{aligned} \tag{3.1.6}$$

for $(a,b,c,z) \in \mathbb{C}^4$ with $|\arg(1-z)| < \pi$, ([JoTh80], p 199). The Nörlund fraction has the form

$$\begin{aligned} c \cdot \frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a+1,b+1;c+1;z)} &= c - (a+b+1)z + \frac{(a+1)(b+1)(z-z^2)}{c+1-(a+b+3)z} \underset{+}{} \\ &\frac{(a+2)(b+2)(z-z^2)}{c+2-(a+b+5)z} \underset{+}{} \frac{(a+3)(b+3)(z-z^2)}{c+3-(a+b+7)z} \underset{+}{} \cdots \end{aligned} \tag{3.1.7}$$

([LoWa92], p 304). $D_c := \{(a,b,c,z) \in \mathbb{C}^4;\ \mathrm{Re}(z) \neq \frac{1}{2}\}$, $D_f := \{(a,b,c,z) \in \mathbb{C}^4;\ \mathrm{Re}(z) < \frac{1}{2}\}$. The Euler fraction has the form

$$\begin{aligned} c \cdot \frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a,b+1;c+1;z)} &= c + (b-a+1)z - \frac{(c-a+1)(b+1)z}{c+1+(b-a+2)z} \underset{-}{} \\ &\frac{(c-a+2)(b+2)z}{c+2+(b-a+3)z} \underset{-}{} \frac{(c-a+3)(b+3)z}{c+3+(b-a+4)z} \underset{-}{} \cdots, \end{aligned} \tag{3.1.8}$$

([LoWa92], p 308). $D_c := \{(a,b,c,z) \in \mathbb{C}^4;\ |z| \neq 1\}$, $D_c := \{(a,b,c,z) \in \mathbb{C}^4;\ |z| < 1,\ (c-a) \neq -1,-2,-3,\dots\}$.

By setting $b := 0$ in (3.1.2), (3.1.5), (3.1.6) or (3.1.7) and using (1.2.1) we get continued fraction expansions for ${}_2F_0(a,1;z)$ and ${}_2F_1(a,1;c+1;z)$. Similarly, $a := 0$ in (3.1.3) or (3.1.4) gives continued fraction expansions for ${}_1F_1(1;c+1;z)$. A different expansion is

$$
\begin{aligned}
{}_2F_1(a,1;c+1;z) = \frac{\Gamma(1-a)\Gamma(c+1)}{\Gamma(c-a+1)} \frac{(1-z)^{c-a}}{(-z)^c} - \\
\frac{c}{1-c+(a-1)z+} \frac{1(1-c)(z-1)}{3-c+(a-2)z+} \frac{2(2-c)(z-1)}{5-c+(a-3)z+\cdots},
\end{aligned}
\tag{3.1.9}
$$

([Bern89], p 164). $D_c := \{(a,c,z) \in \mathbb{C}^3;\ |z-1| \neq 1\}$, $D_f := \{(a,c,z) \in \mathbb{C}^3;\ |z-1| < 1$ and $c \neq 0,1,2,\dots\}$. From this follows after some computation, ([Bern89], p 165) that

$$
\begin{aligned}
{}_1F_1(1;c+1;z) &= \frac{e^z\Gamma(c+1)}{z^c} - \frac{c}{z+} \frac{1-c}{1} \frac{1}{+z+} \frac{2-c}{1} \frac{2}{+z+} \frac{3-c}{1} +\cdots \\
&= \frac{e^z\Gamma(c+1)}{z^c} - \frac{c}{z+1-c-} \frac{1(1-c)}{z+3-c-} \frac{2(2-c)}{z+5-c-} \frac{3(3-c)}{z+7-c-\cdots}
\end{aligned}
\tag{3.1.10}
$$

for $|\arg z| < \pi$, ([Bern89], p 165) (the second continued fraction is the even part of the first one).

A.3.2 Special examples with ${}_0F_1$

The **Bessel function** of the first kind and order ν is

$$
J_\nu(z) := \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k (z/2)^{2k}}{k!\,\Gamma(\nu+k+1)} = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(\nu+1;-\tfrac{z^2}{4}),
\tag{3.2.1}
$$

so that by (3.1.1)

$$
\begin{aligned}
\frac{J_{\nu+1}(z)}{J_\nu(z)} &= \frac{z}{2(\nu+1)} \cdot \frac{{}_0F_1(\nu+2;-z^2/4)}{{}_0F_1(\nu+1;-z^2/4)} \\
&= \frac{z}{2(\nu+1)-} \frac{z^2}{2(\nu+2)-} \frac{z^2}{2(\nu+3)-} \frac{z^2}{2(\nu+4)-\cdots}
\end{aligned}
\tag{3.2.2}
$$

for $z \in \mathbb{C}$, $\nu \neq -1,-2,-3,\dots$, ([JoTh80], p 211).

A.3.3 Special examples with ${}_2F_0$

The connection (see for instance ([Wall48], p 352, p 355))

$$
{}_2F_0(a,b;-z) \sim \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-t}t^{a-1}}{(1+tz)^b}dt = \frac{1}{\Gamma(b)} \int_0^\infty \frac{e^{-t}t^{b-1}}{(1+tz)^a}dt
\tag{3.3.1}
$$

implies that (3.1.2) – (3.1.3) lead to continued fraction expansions for ratios of such integrals. In particular, the **incomplete gamma function** $\Gamma(a,z)$ satisfies

$$
\Gamma(a,z) := \int_z^\infty e^{-t}t^{a-1}dt \sim e^{-z}z^{a-1}\,{}_2F_0(1-a,1;-1/z)\,,
\tag{3.3.2}
$$

([EMOT53], p 266). Hence, by (3.1.2)

$$\begin{aligned}\Gamma(a,z) &= \frac{e^{-z}z^a}{z\,+}\,\frac{1-a}{1\,+}\,\frac{1}{z+}\,\frac{2-a}{1\,+}\,\frac{2}{z+}\,\frac{3-a}{1\,+}\,\frac{3}{z+}\cdots \\ &= \frac{e^{-z}z^a}{1+z-a-}\,\frac{1(1-a)}{3+z-a-}\,\frac{2(2-a)}{5+z-a-}\,\frac{3(3-a)}{7+z-a-}\cdots\end{aligned} \tag{3.3.3}$$

for $(a,z)\in\mathbb{C}^2$ with $|\arg z|<\pi$, ([AbSt65], p 260, p 263), ([Khov63], p 144), where the second continued fraction is the even part of the first one.

This (and the expressions to come) are to be interpreted in the following way: The integral in (3.3.2) is taken for real z. Then $\Gamma(a,z)$ is the analytic continuation of this function to the given domain.

The **complementary error function** erfc z satisfies

$$\operatorname{erfc} z := \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt = \frac{1}{\sqrt{\pi}}\Gamma(\tfrac{1}{2},z^2) \sim \frac{1}{\sqrt{\pi}}e^{-z^2}z^{-1}\,{}_2F_0(\tfrac{1}{2},1;-1/z^2) \tag{3.3.4}$$

([EMOT53], p 266), which means that by (3.1.2)

$$\begin{aligned}\operatorname{erfc} z &= \frac{2}{\sqrt{\pi}}e^{-z^2}\left\{\frac{1}{2z+}\,\frac{2}{2z+}\,\frac{4}{2z+}\,\frac{6}{2z+}\,\frac{8}{2z+}\cdots\right\} \\ &= \frac{2}{\sqrt{\pi}}e^{-z^2}\left\{\frac{z}{1+2z^2-}\,\frac{1\cdot 2}{5+2z^2-}\,\frac{3\cdot 4}{9+2z^2-}\,\frac{5\cdot 6}{13+2z^2-}\,\frac{7\cdot 8}{17+2z^2-}\cdots\right\},\end{aligned} \tag{3.3.5}$$

([JoTh80], p 219). (There is a slightly different notation in ([JoTh80]).) $D_c := \{z\in\mathbb{C};\ \operatorname{Re} z\neq 0\}$, $D_f := \{z\in\mathbb{C};\ \operatorname{Re} z>0\}$. Again the second continued fraction is the even part of the first one. If we integrate this complementary error function we get similar expressions:

$$i^{-1}\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\,e^{-z^2}\ ,\ i^0\operatorname{erfc} z = \operatorname{erfc} z\ ,\ i^n\operatorname{erfc} z = \int_z^\infty i^{n-1}\operatorname{erfc} t\,dt \tag{3.3.6}$$

for $n=1,2,3,\ldots$. Therefore

$$\begin{aligned}\frac{i^{n-1}\operatorname{erfc} z}{i^n\operatorname{erfc} z} &= 2z\frac{{}_2F_0(\frac{n+1}{2},\frac{n}{2};-1/z^2)}{{}_2F_0(\frac{n+1}{2},\frac{n}{2}+1;-1/z^2)} \\ &= 2z+\frac{2(n+1)}{2z\ +}\,\frac{2(n+2)}{2z\ +}\,\frac{2(n+3)}{2z\ +}\cdots\end{aligned} \tag{3.3.7}$$

([JoTh80], p 219). $D_c := \{(n,z)\in\mathbb{C}^2;\ \operatorname{Re} z\neq 0\}$, $D_f := \{(n,z)\in\mathbb{C}^2;\ \operatorname{Re} z>0\}$.

For the **exponential integral**

$$-\operatorname{Ei}(-z) := \int_z^\infty \frac{e^{-t}}{t}\,dt \sim \frac{e^{-z}}{z}\,{}_2F_0(1,1;-\tfrac{1}{z})\,, \tag{3.3.8}$$

([EMOT53], p 267), we get by (3.1.2) and its even part

$$\begin{aligned}\operatorname{Ei}(-\mathrm{z}) &= -\frac{e^{-z}}{z\ +}\,\frac{1}{1+}\,\frac{1}{z+}\,\frac{2}{1+}\,\frac{2}{z+}\,\frac{3}{1+}\,\frac{3}{z+}\,\frac{4}{1+}\cdots \\ &= -\frac{e^{-z}}{1+z-}\,\frac{1^2}{3+z-}\,\frac{2^2}{5+z-}\,\frac{3^2}{7+z-}\,\frac{4^2}{9+z-}\cdots\end{aligned} \tag{3.3.9}$$

for $|\arg z|<\pi$, ([Khov63], p 145).

Similarly, for the **logarithmic integral**

$$\begin{aligned}\operatorname{li} z &:= \int_0^z \frac{dt}{\operatorname{Ln} t} = \operatorname{Ei}(\operatorname{Ln} z) = \frac{z}{\operatorname{Ln} z -}\,\frac{1}{1 -}\,\frac{1}{\operatorname{Ln} z -}\,\frac{2}{1 -}\,\frac{2}{\operatorname{Ln} z -}\cdots \\ &= -\frac{z}{1-\operatorname{Ln} z -}\,\frac{1^2}{3-\operatorname{Ln} z -}\,\frac{2^2}{5-\operatorname{Ln} z -}\,\frac{3^2}{7-\operatorname{Ln} z -}\,\frac{4^2}{9-\operatorname{Ln} z -}\cdots\end{aligned} \tag{3.3.10}$$

for $|\arg(-\operatorname{Ln} z)| < \pi$.

The **plasma dispersion function** is

$$\begin{aligned}P(z) &:= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-z}dt = i\sqrt{\pi}\,e^{-z^2}\operatorname{erfc}(-iz) \\ &= \frac{2z}{1-2z^2 -}\,\frac{1\cdot 2}{5-2z^2 -}\,\frac{3\cdot 4}{9-2z^2 -}\,\frac{5\cdot 6}{13-2z^2 -}\,\frac{7\cdot 8}{17-2z^2 -}\cdots,\end{aligned} \tag{3.3.11}$$

([JoTh80], p 219). $D_c := \{z \in \mathbb{C};\ \operatorname{Im}(z) \neq 0\}$, $D_f := \{z \in \mathbb{C};\ \operatorname{Im}(z) > 0\}$.

$$\frac{\int_0^\infty t^a e^{-bt-t^2/2}dt}{\int_0^\infty t^{a-1} e^{-bt-t^2/2}dt} = \frac{a}{b}\cdot\frac{{}_2F_0\left(\frac{a}{2}, \frac{a+1}{2}; -\frac{1}{b^2}\right)}{{}_2F_0\left(\frac{a}{2}, \frac{a-1}{2}; -\frac{1}{b^2}\right)} = \frac{a}{b +}\,\frac{a+1}{b +}\,\frac{a+2}{b +}\,\frac{a+3}{b +}\cdots, \tag{3.3.12}$$

([Perr57], p 297). $D_c := \{(a,b) \in \mathbb{C}^2;\ \operatorname{Re} b \neq 0\}$, $D_f := \{(a,b) \in \mathbb{C}^2;\ \operatorname{Re} b > 0\}$.

A.3.4 Special examples with ${}_1F_1$

From ([EMOT53], p 255) it follows that

$${}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{tz} t^{a-1}(1-t)^{c-a-1}dt \tag{3.4.1}$$

for $\operatorname{Re}(c) > 0$, $\operatorname{Re}(a) > 0$. Hence (3.1.4), (3.1.5) and (3.1.10) lead to continued fraction expansions of ratios of such integrals.

The **error function** is given by

$$\begin{aligned}\operatorname{erf}(z) &:= \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt = \frac{2}{\sqrt{\pi}} z\, {}_1F_1(\tfrac{1}{2}; \tfrac{3}{2}; -z^2), \quad \text{([EMOT53], p 266)} \\ &= \frac{2}{\sqrt{\pi}} z e^{-z^2}\, {}_1F_1(1; \tfrac{3}{2}; z^2), \quad \text{([JoTh80], p 282).}\end{aligned} \tag{3.4.2}$$

Hence,

$$\begin{aligned}\operatorname{erf}(z) &= \frac{2\,e^{-z^2}}{\sqrt{\pi}}\,\frac{z}{1 -}\,\frac{2z^2}{3 +}\,\frac{4z^2}{5 -}\,\frac{6z^2}{7 +}\,\frac{8z^2}{9 -}\cdots \\ &= \frac{2\,e^{-z^2}}{\sqrt{\pi}}\,\frac{z}{1-2z^2 +}\,\frac{4z^2}{3-2z^2 +}\,\frac{8z^2}{5-2z^2 +}\,\frac{12z^2}{7-2z^2 +}\cdots\end{aligned} \tag{3.4.3}$$

for $z \in \mathbb{C}$, ([JoTh80], p 208 and 282).

The error function is related to **Dawson's integral**

$$\int_0^z e^{t^2}\,dt = \frac{i\sqrt{\pi}}{2}\operatorname{erf}(-iz), \quad \text{([JoTh80], p 208)} \tag{3.4.4}$$

and to the **Fresnel integrals**

$$C(z) := \int_0^z \cos\left(\frac{\pi}{2}t^2\right) dt, \qquad S(z) := \int_0^z \sin\left(\frac{\pi}{2}t^2\right) dt \tag{3.4.5}$$

by

$$\begin{aligned} C(z) + iS(z) &= \int_0^z e^{it^2\pi/2}\, dt = \sqrt{\frac{-2}{i\pi}} \int_0^{\sqrt{-i\pi/2}\cdot z} e^{-u^2} du \\ &= \frac{1+i}{2}\operatorname{erf}\left(\frac{\sqrt{\pi}}{2}(1-i)z\right). \end{aligned} \tag{3.4.6}$$

The **incomplete gamma function**

$$\begin{aligned} \gamma(a,z) := \int_0^z e^{-t}t^{a-1}dt &= \frac{z^a}{a}e^{-z}\,{}_1F_1(1;a+1;z) \\ &= \frac{z^a e^{-z}}{a} \frac{az}{-a+1} \frac{1z}{+a+2} \frac{(a+1)z}{-a+3} \frac{2z}{+a+4} \frac{(a+2)z}{-a+5} + \cdots \\ &= \frac{z^a e^{-z}}{a} \frac{az}{-1+a+z} \frac{(1+a)z}{-2+a+z} \frac{(2+a)z}{-3+a+z} \frac{(3+a)z}{-4+a+z} - \cdots \end{aligned} \tag{3.4.7}$$

for all $(a,z) \in \mathbb{C}^2$, ([JoTh80], p 209), ([Khov63], p 149–150).

The **Coulomb wave function**

$$F_L(\eta,\rho) = \rho^{L+1}e^{-i\rho}C_L(\eta)_1F_1(L+1-i\eta;2L+2;2i\rho) \tag{3.4.8}$$

where $C_L(\eta) = 2^L \exp(-\pi\eta/2)|\Gamma(L+1+i\eta)|/(2L+1)!$ for $\eta \in \mathbb{R}$, $\rho > 0$ and $L \in \mathbb{N} \cup \{0\}$ satisfies

$$\begin{aligned} \frac{F_L(\eta,\rho)}{F_{L-1}(\eta,\rho)} &= \frac{(L+1)(L^2+\eta^2)^{1/2}}{(2L+1)(\eta+L(L+1)/\rho)-} \frac{L(L+2)((L+1)^2+\eta^2)}{(2L+3)(\eta+(L+1)(L+2)/\rho)-} \\ &\quad \frac{(L+1)(L+3)((L+2)^2+\eta^2)}{(2L+5)(\eta+(L+2)(L+3)/\rho)-\cdots}, \end{aligned} \tag{3.4.9}$$

([JoTh80], p 216). $D_c := \{(L,\eta,\rho) \in \mathbb{C}^3;\ \rho \neq 0\}$, $D_f := \{(L,\eta,\rho) \in (\mathbb{N}\cup\{0\})\times\mathbb{C}^2;\ \rho \neq 0\}$.

It is well known that

$$\sum_{k=0}^\infty \frac{(-z)^k}{k!\,(a+k)} = e^{-z}\sum_{k=0}^\infty \frac{z^k}{(a)_{k+1}} = \frac{e^{-z}}{a}\,{}_1F_1(1;a+1;z)\,, \tag{3.4.10}$$

([Bern89], p 166). This means for instance that

$$\sum_{k=0}^\infty \frac{(-z)^k}{k!\,(a+k)} = \frac{\Gamma(a)}{z^a} - \frac{e^{-z}}{z+1-a-} \frac{1(1-a)}{z+3-a-} \frac{2(2-a)}{z+5-a-\cdots} \tag{3.4.11}$$

for $(a,z) \in \mathbb{C}^2$ with $|\arg z| < \pi$, and

$$\sum_{k=0}^\infty \frac{z^k}{1\cdot 3\cdots(2k+1)} = \sqrt{\frac{\pi}{2z}}e^{z/2} - \frac{1}{z+1-} \frac{1\cdot 2}{z+5-} \frac{3\cdot 4}{z+9-} \frac{5\cdot 6}{z+13-\cdots} \tag{3.4.12}$$

for $|\arg z| < \pi$, ([Bern89], p 166), and

$$\begin{aligned} ze^{-z^2}\sum_{k=0}^{\infty}\frac{(2z^2)^k}{1\cdot 3\cdots(2k+1)} &= \int_0^z e^{-t^2}dt \\ &= \frac{\sqrt{\pi}}{2} - \frac{e^{-z^2}}{2z^2+1-}\,\frac{1\cdot 2}{2z^2+5-}\,\frac{3\cdot 4}{2z^2+9-}\,\frac{5\cdot 6}{2z^2+13-}\cdots, \end{aligned} \tag{3.4.13}$$

([Bern89], p 166). $D_c := \{z \in \mathbb{C};\ \operatorname{Re} z \neq 0\}$, $D_f := \{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$.

A.3.5 Special examples with ${}_2F_1$

$$\begin{aligned} \int_0^z \frac{t^p\,dt}{1+t^q} &= \frac{z^{p+1}}{q}\,{}_2F_1\left(\frac{p+1}{q}, 1; 1+\frac{p+1}{q}; -z^q\right) \\ &= \frac{z^{p+1}}{0q+p+1+}\,\frac{(0q+p+1)^2 z^q}{1q+p+1}\,\frac{(1q)^2 z^q}{+2q+p+1+} \\ &\quad \frac{(1q+p+1)^2 z^q}{3q+p+1}\,\frac{(2q)^2 z^q}{+4q+p+1+}\cdots \end{aligned} \tag{3.5.1}$$

for $p, q > 0$ with $|\arg(1+z^q)| < \pi$, ([Khov63], p 126).

Incomplete beta functions are given by

$$B_x(p,q) := \int_0^x t^{p-1}(1-t)^{q-1}dt = \frac{x^p}{p}\,{}_2F_1(p, 1-q; p+1; x) \tag{3.5.2}$$

for $p > 0, q > 0$ and $0 \le x \le 1$, ([EMOT53], p 87). Hence, by (3.1.6) and (3.1.8)

$$\begin{aligned} &\frac{B_x(p+1,q)}{B_x(p,q)} = \frac{px}{p+1}\,\frac{{}_2F_1(p+1, 1-q; p+2; x)}{{}_2F_1(p, 1-q; p+1; x)} \\ &= \frac{px}{p+1-}\,\frac{1(1-q)x}{p+2}\,\frac{(p+1)(p+q+1)x}{-\quad p+3} - \\ &\qquad \frac{2(2-q)x}{p+4}\,\frac{(p+2)(p+q+2)x}{-\quad p+5}\,-\cdots;\quad |\arg(1-x)| < \pi \end{aligned} \tag{3.5.3}$$

for $p > 0$, $q > 0$, ([JoTh80], p 217), and

$$\begin{aligned} \frac{B_x(p+1,q)}{B_x(p,q)} &= \frac{px}{p+1+(p+q)x-}\,\frac{(p+q+1)(p+1)x}{p+2+(p+q+1)x} \\ &\quad \frac{(p+q+2)(p+2)x}{-p+3+(p+q+2)x-}\cdots, \end{aligned} \tag{3.5.4}$$

([JoTh80], p 217). $D_c := \{p > 0,\ q > 0,\ x \in \mathbb{C};\ |x| \neq 1\}$, $D_f := \{(p,q,x) \in D_c;\ |x| < 1\}$.

Legendre functions of the first kind of degree $\alpha \in \mathbb{R}$ and order $m \in \mathbb{N} \cup \{0\}$ are given by

$$\begin{aligned} P_\alpha^m(z) &:= \frac{1}{\pi}\frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)}\int_0^\pi (z+(z^2-1)^{1/2}\cos t)^\alpha \cos mt\,dt \\ &= \frac{1}{\Gamma(1-m)}\left(\frac{z+1}{z-1}\right)^{m/2}{}_2F_1\left(-\alpha, \alpha+1; 1-m; \frac{1-z}{2}\right). \end{aligned} \tag{3.5.5}$$

From ([Gaut67], formula (6.1) on p 55) it follows that

$$\frac{P_\alpha^m(z)}{P_\alpha^{m-1}(z)} = \frac{(m+\alpha)(m-\alpha-1)}{-\frac{2mz}{(z^2-1)^{1/2}}} - \frac{(m+1+\alpha)(m-\alpha)}{-\frac{2(m+1)z}{(z^2-1)^{1/2}}} - \frac{(m+2+\alpha)(m+1-\alpha)}{-\frac{2(m+2)z}{(z^2-1)^{1/2}}} - \cdots \tag{3.5.6}$$

$$\sim -\frac{(m+\alpha)(m-\alpha-1)\sqrt{z^2-1}}{2mz} - \frac{(m+1+\alpha)(m-\alpha)(z^2-1)}{2(m+1)z} - \frac{(m+2+\alpha)(m+1-\alpha)(z^2-1)}{2(m+2)z} - \frac{(m+3+\alpha)(m+2-\alpha)(z^2-1)}{2(m+3)z} - \cdots$$

$D_c := \{(\alpha,m,z)\in\mathbb{C}^3;\ \mathrm{Re}(z)\neq 0\}$, $D_f := \{(\alpha,m,z)\in\mathbb{R}\times(\mathbb{N}\cup\{0\})\times\mathbb{C};\ \mathrm{Re}(z)>0\}$.

Legendre functions of the second kind of degree $\alpha\in\mathbb{R}$ and order $m\in\mathbb{N}\cup\{0\}$ are given by

$$Q_\alpha^m(z) := (-1)^m \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-m+1)} \int_0^\infty \frac{\cosh mt}{(z+(z^2-1)^{1/2}\cosh t)^{\alpha+1}}\,dt$$
$$= \frac{\sqrt{\pi}\,e^{im\pi}}{(2z)^{\alpha+1}}\left(1-\frac{1}{z^2}\right)^{m/2} \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+m+\frac{3}{2})} \cdot {}_2F_1\left(\tfrac{\alpha+m+2}{2}, \tfrac{\alpha+m+1}{2}; \alpha+\tfrac{3}{2}; 1/z^2\right). \tag{3.5.7}$$

In ([JoTh80], p 205) it is proved that

$$\frac{Q_\alpha^m(z)}{Q_{\alpha+1}^m(z)} = \frac{1}{\alpha+m+1}\left\{(2\alpha+3)z - \frac{(\alpha+m+2)^2}{(2\alpha+5)z} - \frac{(\alpha+m+3)^2}{(2\alpha+7)z} - \frac{(\alpha+m+4)^2}{(2\alpha+9)z} - \frac{(\alpha+m+5)^2}{(2\alpha+11)z} - \frac{(\alpha+m+6)^2}{(2\alpha+13)z} - \cdots\right\}. \tag{3.5.8}$$

$D_c := \{(\alpha,m,z)\in\mathbb{C}^3;\ z\notin[-1,1]\}$, $D_f := \{(\alpha,m,z)\in\mathbb{R}\times(\mathbb{N}\cup\{0\})\times\mathbb{C};\ z\notin[-1,1]\}$.

A.3.6 Some integrals

Hypergeometric functions can be written in terms of integrals. This has already been used to some extent in the preceding subsections, and we refer to ([AbSt65]) and ([EMOT53]) for further details. Here we shall just list some simple examples without bringing in the hypergeometric functions themselves.

$$\int_0^1 x^s e^{1-x}\,dx = \frac{1}{s+}\frac{1}{s+1+}\frac{1}{s+1+}\frac{1}{s+1+}\cdots = \sum_{n=1}^\infty \frac{1}{(s+1)_n};\ s\in\mathbb{C}, \tag{3.6.1}$$

([Khru06b]).

$$\int_0^1 \frac{x^s}{1+x^2}\,dx = \frac{1}{s+}\frac{1^2}{s+}\frac{2^2}{s+}\frac{3^2}{s+}\cdots = \sum_{k=0}^\infty \frac{2\cdot(-1)^k}{s+2k+1} = \sum_{k=1}^\infty \frac{4}{(s+2k)^2-1}, \tag{3.6.2}$$

([Khru06b]). $D_c := \{s \in \mathbb{C};\ \mathrm{Re}\, s \neq 0\}$, $D_f := \{s \in \mathbb{C};\ \mathrm{Re}\, s > 0\}$.

$$\int_0^\infty \frac{e^{-t}dt}{t+z} = \frac{1}{z+1} \underset{-}{} \frac{1^2}{z+3} \underset{-}{} \frac{2^2}{z+5} \underset{-}{} \frac{3^2}{z+7} \underset{-}{} \cdots ; \quad |\arg z| < \pi, \tag{3.6.3}$$

([BoSh89], p 20).

$$\begin{aligned} &\int_0^\infty \frac{e^{-t/z}}{(1+t)^n} dt \\ &= \frac{z}{1} \underset{+}{} \frac{nz}{1} \underset{+}{} \frac{1z}{1} \underset{+}{} \frac{(n+1)z}{1} \underset{+}{} \frac{2z}{1} \underset{+}{} \frac{(n+2)z}{1} \underset{+}{} \frac{3z}{1} \underset{+}{} \cdots \\ &= \frac{z}{1+nz} \underset{-}{} \frac{nz^2}{1+(n+2)z} \underset{-}{} \frac{2(n+1)z^2}{1+(n+4)z} \underset{-}{} \frac{3(n+2)z^2}{1+(n+6)z} \underset{-}{} \cdots \end{aligned} \tag{3.6.4}$$

for $(n, z) \in \mathbb{R} \times \mathbb{C}$ with $|\arg z| < \pi$, ([BoSh89], p 157). The second continued fraction is the even part of the first one.

$$\int_0^\infty \frac{e^{-tz}}{\cosh^2 t} dt = \frac{1}{z} \underset{+}{} \frac{1 \cdot 2}{z} \underset{+}{} \frac{2 \cdot 3}{z} \underset{+}{} \frac{3 \cdot 4}{z} \underset{+}{} \cdots = 2z \sum_{k=0}^\infty \frac{(-1)^k}{(z+2k)(z+2k+2)}, \tag{3.6.5}$$

([Khru06b]). $D_c := \{z \in \mathbb{C};\ \mathrm{Re}\, z \neq 0\}$, $D_f := \{z \in \mathbb{C};\ \mathrm{Re}\, z > 0\}$.

For **Jacobi's elliptic functions** $\mathrm{sn}\, t$, $\mathrm{cn}\, t$ and $\mathrm{dn}\, t$ with modulus k we have

$$\int_0^\infty e^{-tz} \mathrm{sn}\, t\, dt = \frac{1}{1^2(1+k^2)+z^2} \underset{-}{} \frac{1 \cdot 2^2 \cdot 3k^2}{3^2(1+k^2)+z^2} \underset{-}{} \frac{3 \cdot 4^2 \cdot 5k^2}{5^2(1+k^2)+z^2} \underset{-}{} \cdots, \tag{3.6.6}$$

([Wall48], p 374). $D_c := \{(k, z) \in \mathbb{C}^2;\ |k| \neq 1\}$, $D_f := \{(k, z) \in \mathbb{C}^2;\ |k| < 1\}$,

$$\int_0^\infty e^{-tz} \mathrm{sn}^2 t\, dt = \frac{2}{2^2(1+k^2)+z^2} \underset{-}{} \frac{2 \cdot 3^2 \cdot 4k^2}{4^2(1+k^2)+z^2} \underset{-}{} \frac{4 \cdot 5^2 \cdot 6k^2}{6^2(1+k^2)+z^2} \underset{-}{} \cdots, \tag{3.6.7}$$

([Wall48], p 375). $D_c := \{(k, z) \in \mathbb{C}^2;\ |k| \neq 1\}$, $D_f := \{(k, z) \in \mathbb{C}^2;\ |k| < 1\}$,

$$\int_0^\infty e^{-tz} \mathrm{cn}\, t\, dt = \frac{1}{z} \underset{+}{} \frac{1^2}{z} \underset{+}{} \frac{2^2 k^2}{z} \underset{+}{} \frac{3^2}{z} \underset{+}{} \frac{4^2 k^2}{z} \underset{+}{} \frac{5^2}{z} \underset{+}{} \cdots, \tag{3.6.8}$$

([Perr57], p 220). $D_c := \{(k, z) \in \mathbb{R} \times \mathbb{C};\ \mathrm{Re}\, z \neq 0\}$, $D_f := \{(k, z) \in \mathbb{R} \times \mathbb{C};\ \mathrm{Re}\, z > 0\}$,

$$\int_0^\infty e^{-tz} \mathrm{dn}\, t\, dt = \frac{1}{z} \underset{+}{} \frac{1^2 k^2}{z} \underset{+}{} \frac{2^2}{z} \underset{+}{} \frac{3^2 k^2}{z} \underset{+}{} \frac{4^2}{z} \underset{+}{} \frac{5^2 k^2}{z} \cdots, \tag{3.6.9}$$

([Wall48], p 374). $D_c := \{(k, z) \in \mathbb{R} \times \mathbb{C};\ \mathrm{Re}\, z \neq 0\}$, $D_f := \{(k, z) \in \mathbb{R} \times \mathbb{C};\ \mathrm{Re}\, z > 0\}$, and

$$\begin{aligned} &\int_0^\infty \frac{\mathrm{sn}\, t\ \mathrm{cn}\, t}{\mathrm{dn}\, t} e^{-tz} dt = \\ &\frac{1}{2 \cdot 1^2(2-k^2)+z^2} \underset{-}{} \frac{1 \cdot 2^2 \cdot 3k^4}{2 \cdot 3^2(2-k^2)+z^2} \underset{-}{} \frac{3 \cdot 4^2 \cdot 5k^4}{2 \cdot 5^2(2-k^2)+z^2} \underset{-}{} \cdots \end{aligned} \tag{3.6.10}$$

for $(k,z) \in \mathbb{C}^2$ with $|1-k^2|<1$, ([Wall48], p 375).

$$\int_0^\infty \left(\frac{1-c}{e^{t(1-c)}-c^b}\right)^a e^{-tz}dt = \frac{r^a}{z+}\frac{ar}{1+}\frac{rc^b}{z+}\frac{(a+1)r}{1+}\frac{2rc^b}{z+}\frac{(a+2)r}{1+}\cdots \tag{3.6.11}$$

where $r := (1-c)/(1-c^b)$, for $(a,b,c,z) \in \mathbb{C}^4$ with $a>0$, $c^b>0$ and $|\arg(r/z)|<\pi$, ([Wall48], p 359).

$$\int_0^\infty \frac{te^{-tz}}{\sinh t}dt = \frac{1}{z+}\frac{1^4}{3z+}\frac{2^4}{5z+}\frac{3^4}{7z+}\cdots, \tag{3.6.12}$$

([Wall48], p 371). $D_c := \{z \in \mathbb{C};\ \operatorname{Re} z \neq 0\}$, $D_f := \{z \in \mathbb{C};\ \operatorname{Re} z > 0\}$.

$$\begin{aligned}\int_0^\infty \frac{2te^{-tz}}{e^t+e^{-t}}dt &= \int_0^\infty \frac{te^{-tz}}{\cosh t}dt = 2\sum_{n=0}^\infty \frac{(-1)^n}{(z+1+2n)^2}\\ &= \frac{1}{z^2-1+}\frac{4\cdot 1^2}{1+}\frac{4\cdot 1^2}{z^2-1+}\frac{4\cdot 2^2}{1+}\frac{4\cdot 2^2}{z^2-1+}\frac{4\cdot 3^2}{1+}\cdots,\end{aligned} \tag{3.6.13}$$

([Perr57], p 30). $D_c := \{z \in \mathbb{C};\ |\arg(z^2-1)|<\pi\}$, $D_f := \{z \in \mathbb{C};\ \operatorname{Re} z > 0$ and $z \notin (0,1]\}$. For instance, for $z =: \sqrt{5}$ we get

$$\int_0^\infty \frac{4te^{-\sqrt{5}t}}{\cosh t}dt = \frac{1}{1+}\frac{1^2}{1+}\frac{1^2}{1+}\frac{2^2}{1+}\frac{2^2}{1+}\frac{3^2}{1+}\frac{3^2}{1+}\cdots, \text{ ([Perr57], p 30).} \tag{3.6.14}$$

A.3.7 Gamma function expressions by Ramanujan

Ramanujan produced quite a number of continued fraction expansions of ratios of gamma functions. These ratios have all proved to be connected to hypergeometric functions, ([Rama57], [Bern89]). We use Ramanujan's notation

$$\begin{aligned}\prod_\varepsilon \Gamma(a+\varepsilon b+c) &:= \Gamma(a+b+c)\Gamma(a-b+c),\\ \prod_\varepsilon \Gamma(a+\varepsilon b+\varepsilon c+d) &:= \Gamma(a+b+c+d)\Gamma(a-b+c+d)\times\\ &\quad\times\Gamma(a+b-c+d)\Gamma(a-b-c+d)\end{aligned} \tag{3.7.1}$$

and so on. That is, $\varepsilon = \pm 1$, and the product is taken over all different combinations of the εs.

$$\begin{aligned}&\frac{1-R}{1+R} = \frac{p}{z+}\frac{1^2-q^2}{z+}\frac{2^2-p^2}{z+}\frac{3^2-q^2}{z+}\frac{4^2-p^2}{z+}\cdots\\ &\text{where } R = \prod_\varepsilon \frac{\Gamma\left(\frac{z+p+\varepsilon q+1}{4}\right)}{\Gamma\left(\frac{z+p+\varepsilon q+3}{4}\right)}\cdot\prod_\varepsilon \frac{\Gamma\left(\frac{z-p+\varepsilon q+3}{4}\right)}{\Gamma\left(\frac{z-p+\varepsilon q+1}{4}\right)},\end{aligned} \tag{3.7.2}$$

([Bern89], p 156). $D_c := \{(p,q,z) \in \mathbb{C}^3;\ \operatorname{Re} z \neq 0\}$, $D_f := \{(p,q,z) \in \mathbb{C}^3;\ \operatorname{Re} z > 0\}$.

From this it follows that

$$\begin{aligned}&\sum_{k=1}^{\infty}\left\{\frac{(-1)^{k+1}}{z+q+2k-1}+\frac{(-1)^{k+1}}{z-q+2k-1}\right\}\\&=\int_0^{\infty}\frac{\cosh(qt)e^{-tz}}{\cosh t}dt=\frac{1}{z+}\frac{1^2-q^2}{z}\,+\frac{2^2}{z+}\frac{3^2-q^2}{z}\,+\frac{4^2}{z+}\cdots,\end{aligned}\tag{3.7.3}$$

([Bern89], p 148), $D_c := \{(q,z)\in\mathbb{C}^2;\ \operatorname{Re} z\neq 0\}$, $D_f := \{(q,z)\in\mathbb{C}^2;\ \operatorname{Re} z>0\}$, and

$$\tanh\left\{\int_0^{\infty}\frac{\sinh(at)e^{-tz}}{t\cosh t}dt\right\}=\frac{a}{z+}\frac{1^2}{z+}\frac{2^2-a^2}{z}\,+\frac{3^2}{z+}\frac{4^2-a^2}{z}\,+\cdots,\tag{3.7.4}$$

([Wall48], p 372), $D_c := \{(a,z)\in\mathbb{C}^2;\ \operatorname{Re} z\neq 0\}$, $D_f := \{(a,z)\in\mathbb{C}^2;\ \operatorname{Re} z>0\}$, and

$$\begin{aligned}&\tanh\left\{\frac{1}{2}\int_0^{\infty}\frac{\sinh(2at)e^{-tz}}{t\cosh t}dt\right\}=\\&\frac{a}{z+}\frac{1^2-a^2}{z}\,+\frac{2^2-a^2}{z}\,+\frac{3^2-a^2}{z}\,+\frac{4^2-a^2}{z}\,+\cdots,\end{aligned}\tag{3.7.5}$$

([Wall48], p 371), $D_c := \{(a,z)\in\mathbb{C}^2;\ \operatorname{Re} z\neq 0\}$, $D_f := \{(a,z)\in\mathbb{C}^2;\ \operatorname{Re} z>0\}$.

Solving (3.7.2) for $1/R$ gives

$$\frac{1}{R}=1+\frac{2p}{z-p+}\frac{1^2-q^2}{z}\,+\frac{2^2-p^2}{z}\,+\frac{3^2-q^2}{z}\,+\frac{4^2-p^2}{z}\,+\cdots,\tag{3.7.6}$$

([Perr57], p 34). $D_c := \{(p,z)\in\mathbb{C}^2;\ \operatorname{Re} z\neq 0\}$, $D_f := \{(p,z)\in\mathbb{C}^2;\ \operatorname{Re} z>0\}$. The values $p := q := 1/2$ lead to

$$\frac{z}{4}\frac{\Gamma^2\left(\frac{z}{4}\right)}{\Gamma^2\left(\frac{z+2}{4}\right)}=1+\frac{2}{2z-1+}\frac{1\cdot 3}{2z}\,+\frac{3\cdot 5}{2z}\,+\frac{5\cdot 7}{2z}\,+\cdots\quad\text{for }\operatorname{Re}(z)>0\tag{3.7.7}$$

and thus, for $z := 4n$ or $z := 4n-2$ where $n\in\mathbb{N}$, we have

$$\frac{1}{n\pi}\left(\frac{2\cdot 4\cdot\cdots\cdot(2n)}{1\cdot 3\cdot\cdots\cdot(2n-1)}\right)^2=1+\frac{2}{8n-1+}\frac{1\cdot 3}{8n}\,+\frac{3\cdot 5}{8n}\,+\frac{5\cdot 7}{8n}\,+\cdots,\tag{3.7.8}$$

([Perr57], p 34),

$$\frac{2n^2\pi}{2n-1}\left(\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n)}\right)^2=1+\frac{2}{8n-5+}\frac{1\cdot 3}{8n-4+}\frac{3\cdot 5}{8n-4+}\frac{5\cdot 7}{8n-4+}\cdots\tag{3.7.9}$$

for $n\in\mathbb{N}$, ([Perr57], p 34).

$$\frac{a+1}{a}\frac{\int_0^1 t^a\left(\frac{1-t}{1+t}\right)^b dt}{\int_0^1 t^{a-1}\left(\frac{1-t}{1+t}\right)^b dt}=\frac{a+1}{2b}\,+\frac{(a+1)(a+2)}{2b}\,+\frac{(a+2)(a+3)}{2b}\,+\cdots,\tag{3.7.10}$$

([Perr57], p 299). $D_c := \{(a,b) \in \mathbb{C}^2;\ \mathrm{Re}(b) \neq 0\}$, $D_f := \{(a,b) \in \mathbb{C}^2;\ \mathrm{Re}(b) > 0\}$. From this follows directly that also

$$\frac{\displaystyle\int_0^1 t^a \left(\frac{1-t}{1+t}\right)^b \frac{dt}{1-t}}{\displaystyle\int_0^1 t^a \left(\frac{1-t}{1+t}\right)^b \frac{dt}{1-t^2}} = 1 + \frac{a+1}{2b} \, {}_+ \, \frac{(a+1)(a+2)}{2b} \, {}_+ \, \frac{(a+2)(a+3)}{2b} \, {}_+ \cdots, \tag{3.7.11}$$

([Perr57], p 300). $D_c := \{(a,b) \in \mathbb{C}^2;\ \mathrm{Re}(b) \neq 0\}$, $D_f := \{(a,b) \in \mathbb{C}^2;\ \mathrm{Re}(b) > 0\}$. A formula of the same character as (3.7.2) is

$$\begin{aligned} &\prod_\varepsilon \left(\Gamma\Big(\frac{z+\varepsilon p+\varepsilon q+1}{4}\Big)\right) \Big/ \left(\Gamma\Big(\frac{z+\varepsilon p+\varepsilon q+3}{4}\Big)\right) = \\ &\frac{8}{\frac{1}{2}(z^2-p^2+q^2-1)} \, {}_+ \, \frac{1^2-q^2}{1} \, {}_+ \, \frac{1^2-p^2}{z^2-1} \, {}_+ \, \frac{3^2-q^2}{1} \, {}_+ \, \frac{3^2-p^2}{z^2-1} \, {}_+ \cdots, \end{aligned} \tag{3.7.12}$$

([Bern89], p 159). $D_c := \{(p,q,z) \in \mathbb{C}^3;\ |\arg(z^2-1)| < \pi\}$, $D_f := \{(p,q,z) \in \mathbb{C}^3;\ \mathrm{Re}\, z > 0\} \setminus \{(p,q,z) \in \mathbb{C}^3;\ 0 < z \leq 1\}$.

$$\prod_\varepsilon \frac{\Gamma\left(\dfrac{z+\varepsilon q+1}{4}\right)}{\Gamma\left(\dfrac{z+\varepsilon q+3}{4}\right)} = \frac{4}{z} \, {}_+ \, \frac{1^2-q^2}{2z} \, {}_+ \, \frac{3^2-q^2}{2z} \, {}_+ \, \frac{5^2-q^2}{2z} \, {}_+ \cdots, \tag{3.7.13}$$

([Bern89], p 140). $D_c := \{(q,z) \in \mathbb{C}^2;\ \mathrm{Re}\, z \neq 0\}$, $D_f := \{(q,z) \in \mathbb{C}^2;\ \mathrm{Re}\, z > 0\}$. For $q := 0$ and $z := 4n-1$ or $z := 4n+1$ for an $n \in \mathbb{N}$, this reduces to

$$4\pi n^2 \left(\frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot \cdots \cdot (2n)}\right)^2 = 4n-1 + \frac{1^2}{8n-2} \, {}_+ \, \frac{3^2}{8n-2} \, {}_+ \, \frac{5^2}{8n-2} \, {}_+ \cdots, \tag{3.7.14}$$

([Perr57], p 36), or

$$\frac{1}{\pi}\left(\frac{2n+1}{n+1}\right)^2 \left(\frac{2 \cdot 4 \cdot \cdots \cdot (2n+2)}{1 \cdot 3 \cdot \cdots \cdot (2n+1)}\right)^2 = 4n+1 + \frac{1^2}{8n+2} \, {}_+ \, \frac{3^2}{8n+2} \, {}_+ \, \frac{5^2}{8n+2} \, {}_+ \cdots, \tag{3.7.15}$$

([Perr57], p 36).

A formula closely related to (3.7.13) is

$$\begin{aligned} &\exp\left\{\int_0^\infty \left(1 - \frac{\cosh 2at}{\cosh 2t}\right) e^{-tz} \frac{dt}{t}\right\} = \\ &\qquad 1 + \frac{2(1^2-a^2)}{z^2} \, {}_+ \, \frac{3^2-a^2}{1} \, {}_+ \, \frac{5^2-a^2}{z^2} \, {}_+ \cdots, \end{aligned} \tag{3.7.16}$$

([Wall48], p 371). $D_c := \{(a,z) \in \mathbb{C}^2;\ \mathrm{Re}\, z \neq 0\}$, $D_f := \{(a,z) \in \mathbb{C}^2;\ \mathrm{Re}\, z > 0\}$.

The most involved of Ramanujan's formulas of this type is

$$\frac{R-Q}{R+Q} = \frac{8abcdh}{1\{2S_4-(S_2-2\cdot 0\cdot 1)^2-4(0^2+0+1)^2\}+} \frac{64(a^2-1^2)(b^2-1^2)(c^2-1^2)(d^2-1^2)(h^2-1^2)}{3\{2S_4-(S_2-2\cdot 1\cdot 2)^2-4(1^2+1+1)^2\}}+ \frac{64(a^2-2^2)(b^2-2^2)(c^2-2^2)(d^2-2^2)(h^2-2^2)}{5\{2S_4-(S_2-2\cdot 2\cdot 3)^2-4(2^2+2+1)^2\}}+\cdots, \tag{3.7.17}$$

where $S_4 := a^4+b^4+c^4+d^4+h^4+1$, $S_2 := a^2+b^2+c^2+d^2+h^2-1$, and

$$R := \prod_{\varepsilon}\Gamma\left(\frac{a+\varepsilon(b+c)+\varepsilon(d+h)+1}{2}\right)\cdot\prod_{\varepsilon}\Gamma\left(\frac{a+\varepsilon(b+d)+\varepsilon(c+h)+1}{2}\right),$$
$$Q := \prod_{\varepsilon}\Gamma\left(\frac{a+\varepsilon(b-c)+\varepsilon(d+h)+1}{2}\right)\cdot\prod_{\varepsilon}\Gamma\left(\frac{a+\varepsilon(b+c)+\varepsilon(d-h)+1}{2}\right), \tag{3.7.18}$$

([Bern89], p 163). The expansion (3.7.17) only holds if the continued fraction terminates.

$$\frac{1-R}{1+R} = \frac{2abc}{z^2-a^2-b^2-c^2+1+} \frac{4(a^2-1^2)(b^2-1^2)(c^2-1^2)}{3(z^2-a^2-b^2-c^2+5)}+ \frac{4(a^2-2^2)(b^2-2^2)(c^2-2^2)}{5(z^2-a^2-b^2-c^2+13)}+\cdots \quad \text{for } (a,b,c,z)\in\mathbb{C}^4 \quad \text{where}$$
$$R := \prod_{\varepsilon}\frac{\Gamma\left(\frac{z+a+\varepsilon(b+c)+1}{2}\right)}{\Gamma\left(\frac{z-a+\varepsilon(b+c)+1}{2}\right)}\cdot\prod_{\varepsilon}\frac{\Gamma\left(\frac{z-a+\varepsilon(b-c)+1}{2}\right)}{\Gamma\left(\frac{z+a+\varepsilon(b-c)+1}{2}\right)}, \tag{3.7.19}$$

([Bern89], p 157). The last number in each partial denominator of the continued fraction (i.e., 1, 5, 13, ...) is the number $2n^2+2n+1$ for $n = 0, 1, 2, \ldots$.)

$$\frac{1-R}{1+R} = \frac{ab}{z+} \frac{(a^2-1^2)(b^2-1^2)}{3z}+ \frac{(a^2-2^2)(b^2-2^2)}{5z}+ \frac{(a^2-3^2)(b^2-3^2)}{7z}+\cdots \tag{3.7.20}$$
$$\text{where } R = \prod_{\varepsilon}\Gamma\Big(\frac{z+\varepsilon(a-b)+1}{2}\Big)\Big/\prod_{\varepsilon}\Gamma\Big(\frac{z+\varepsilon(a+b)+1}{2}\Big),$$

([Bern89], p 155). $D_c := \{(a,b,z)\in\mathbb{C}^3;\ \operatorname{Re} z \neq 0\}$, $D_f := \{(a,b,z)\in\mathbb{C}^3;\ \operatorname{Re} z > 0\}$. In particular

$$\sum_{k=0}^{\infty}\left\{\frac{1}{z-a+2k+1}-\frac{1}{z+a+2k+1}\right\} = \lim_{b\to 0}\frac{1}{b}\frac{1-R}{1+R} = \frac{a}{z+} \frac{1^2(1^2-a^2)}{3z}+ \frac{2^2(2^2-a^2)}{5z}+ \frac{3^2(3^2-a^2)}{7z}+\cdots \tag{3.7.21}$$

for $\operatorname{Re}(z) > 0$, ([Bern89], p 149).

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(a+k)(b+k)} = \frac{1}{(a+1)(b+1)+} \frac{(a+1)^2(b+1)^2}{a+b+3}+ \frac{(a+2)^2(b+2)^2}{a+b+5}+\cdots \tag{3.7.22}$$

for $(a,b)\in\mathbb{C}^2$ with $b\neq -1$ if $a\in(-\mathbb{N})$ and $a\neq -1$ if $b\in(-\mathbb{N})$, ([Bern89], p 123).

$$\frac{1-R}{1+R}=\frac{ab}{z^2-1-a^2+}\,\frac{2^2-b^2}{1}\,\frac{2^2-a^2}{+\;z^2-1\;+}\,\frac{4^2-b^2}{1}\,\frac{4^2-a^2}{+\;z^2-1\;+\cdots}\,;$$
$$R:=\prod_{\varepsilon}\frac{\Gamma\Big(\frac{z+\varepsilon(a+b)+3}{4}\Big)}{\Gamma\Big(\frac{z+\varepsilon(a+b)+1}{4}\Big)}\Big/\prod_{\varepsilon}\frac{\Gamma\Big(\frac{z+\varepsilon(a-b)+3}{4}\Big)}{\Gamma\Big(\frac{z+\varepsilon(a-b)+1}{4}\Big)}\,, \tag{3.7.23}$$

([Bern89], p 158). $D_c:=\{(a,b,z)\in\mathbb{C}^3;\ |\arg(z^2-1)|<\pi\}$, $D_f:=\{(a,b,z)\in\mathbb{C}^3;\ \operatorname{Re}z>0\}\setminus\{(a,b,z)\in\mathbb{C}^3;\ 0<z\leq 1\}$. Dividing (3.7.23) by a and letting $a\longrightarrow 0$ in this equality gives

$$\sum_{k=0}^{\infty}\left\{\frac{(-1)^k}{z-b+2k+1}-\frac{(-1)^k}{z+b+2k+1}\right\}=\int_0^{\infty}e^{-tz}\frac{\sinh(bt)}{\cosh t}dt$$
$$=\frac{b}{z^2-1+}\,\frac{2^2-b^2}{1}\,\frac{2^2}{+\,z^2-1+}\,\frac{4^2-b^2}{1}\,\frac{}{+\cdots}\,, \tag{3.7.24}$$

([Bern89], p 150). $D_c:=\{(b,z)\in\mathbb{C}^2;\ |\arg(z^2-1)|<\pi\}$, $D_f:=\{(b,z)\in\mathbb{C}^2;\ \operatorname{Re}z>0\}\setminus\{(b,z)\in\mathbb{C}^2;\ 0<z\leq 1\}$. Of course, dividing this again by b and letting $b\longrightarrow 0$ gives

$$2\sum_{k=0}^{\infty}\frac{(-1)^k}{(z+2k+1)^2}=\frac{1}{z^2-1+}\,\frac{2^2}{1}\,\frac{2^2}{+\,z^2-1+}\,\frac{4^2}{1}\,\frac{4^2}{+\,z^2-1+\cdots}\,, \tag{3.7.25}$$

([Bern89], p 151). $D_c:=\{z\in\mathbb{C};\ |\arg(z^2-1)|<\pi\}$, $D_f:=\{z\in\mathbb{C};\ \operatorname{Re}z>0\}\setminus\{z\in\mathbb{C};\ 0<z\leq 1\}$.

$$1+2z\sum_{k=1}^{\infty}\frac{(-1)^k}{z+2k}=\frac{1}{z+}\,\frac{1\cdot 2}{z}\,\frac{2\cdot 3}{+\;z}\,\frac{3\cdot 4}{+\;z\;+\cdots}\,, \tag{3.7.26}$$

([Bern89], p 151). $D_c:=\{z\in\mathbb{C};\ \operatorname{Re}z\neq 0\}$, $D_f:=\{z\in\mathbb{C};\ \operatorname{Re}z>0\}$.

$$1+2z^2\sum_{k=1}^{\infty}\frac{(-1)^k}{(z+k)^2}=\frac{1}{z+}\,\frac{1^2}{z+}\,\frac{1\cdot 2}{z}\,\frac{2^2}{+\,z+}\,\frac{2\cdot 3}{z}\,\frac{3^2}{+\,z+\cdots}\,, \tag{3.7.27}$$

([Bern89], p 152). $D_c:=\{z\in\mathbb{C};\ \operatorname{Re}z\neq 0\}$, $D_f:=\{z\in\mathbb{C};\ \operatorname{Re}z>0\}$.

$$c\int_0^{\infty}\frac{\sinh at\sinh bt}{\sinh ct}e^{-tz}dt=$$
$$\frac{ab}{1(z^2+c^2-a^2-b^2)-}\,\frac{4\cdot 1^2(1^2c^2-a^2)(1^2c^2-b^2)}{3(z^2+5c^2-a^2-b^2)}\,-$$
$$\frac{4\cdot 2^2(2^2c^2-a^2)(2^2c^2-b^2)}{5(z^2+13c^2-a^2-b^2)}\,-\cdots\,, \tag{3.7.28}$$

where the coefficients for c^2 are $2k^2+2k+1$ in the denominators ([Wall48], p 370). $D_c:=\{(a,b,c,z)\in\mathbb{C}^4;\ \operatorname{Re}\frac{z}{c}\neq 0\}$, $D_f:=\{(a,b,c,z)\in\mathbb{R}^4;\ \operatorname{Re}\frac{z}{c}>0$ and $\operatorname{Re}(z+c-a-b)>0\}$.

$$c\int_0^{\infty}\frac{\sinh at}{\sinh ct}e^{-tz}dt=\frac{a}{z+}\,\frac{1^2(1^2c^2-a^2)}{3z}\,\frac{2^2(2^2c^2-a^2)}{+\quad 5z\quad+\cdots}\,, \tag{3.7.29}$$

([Wall48], p 370). $D_c := \{(a,c,z) \in \mathbb{C}^3;\ \mathrm{Re}(z/c) \neq 0\}$, $D_f := \{(a,c,z) \in \mathbb{C}^3;\ \mathrm{Re}(z/c) > 0\}$.

$$\int_0^\infty \frac{e^{-tz}dt}{(\cosh t + a\sinh t)^b} = \frac{1}{z+ab+}\,\frac{1\cdot b(1-a^2)}{z+a(b+2)+}\,\frac{2(b+1)(1-a^2)}{z+a(b+4)}+\frac{3(b+2)(1-a^2)}{z+a(b+6)}+\cdots, \tag{3.7.30}$$

([Wall48], p 369). $D_c := \{(a,b,z) \in \mathbb{C}^3;\ \mathrm{Re}\,a \neq 0\}$, $D_f := \{(a,b,z) \in \mathbb{C}^3;\ \mathrm{Re}\,a > 0 \text{ and } \mathrm{Re}(b+z) > 0\}$.

$$\int_0^\infty {}_2F_1\Big(a,b;\frac{a+b+1}{2};-\sinh^2 t\Big)e^{-tz}dt = \frac{1}{z+}\,\frac{4\cdot 1ab}{(a+b+1)z+} + \frac{4\cdot 2(a+1)(b+1)(a+b)}{(a+b+3)z} + \frac{4\cdot 3(a+2)(b+2)(a+b+1)}{(a+b+5)z}+\cdots \tag{3.7.31}$$

([Wall48], p 370). $D_c := \{(a,b,z) \in \mathbb{C}^3;\ \mathrm{Re}\,z \neq 0\}$, $D_f := \{(a,b,z) \in \mathbb{C}^3;\ \mathrm{Re}\,z > 0\}$.

$$\begin{aligned}\zeta(3,z+1) &:= \sum_{k=1}^\infty \frac{1}{(z+k)^3}\\ &= \frac{1}{2(z^2+z)+}\,\frac{1^3}{1}+\frac{1^3}{6(z^2+z)+}\,\frac{2^3}{1}+\frac{2^3}{10(z^2+z)}+\cdots,\end{aligned} \tag{3.7.32}$$

([Bern89], p. 153). $D_c := \{z \in \mathbb{C};\ |\arg(z^2+z)| < \pi\} = \{z \in \mathbb{C};\ \mathrm{Re}\,z \neq -\frac{1}{2}\} \setminus [-1,0]$, $D_f := \{z \in \mathbb{C};\ \mathrm{Re}\,z > -\frac{1}{2}\} \setminus [-\frac{1}{2},0]$. The even part of this continued fraction is

$$\zeta(3,z+1) = \frac{1}{1(2z^2+2z+1)-}\,\frac{1^6}{3(2z^2+2z+3)-}\,\frac{2^6}{5(2z^2+2z+7)-}\,\frac{3^6}{7(2z^2+2z+13)-}\cdots. \tag{3.7.33}$$

$D_c := \{z \in \mathbb{C};\ \mathrm{Re}(z^2+\frac{1}{2}) \neq 0\}$, $D_f := \{z \in \mathbb{C};\ \mathrm{Re}(z^2+\frac{1}{2}) > 0\}$. (The numbers 1, 3, 7, 13, … in the denominators are n^2+n+1 for $n = 0, 1, 2, \ldots$.)

$$\begin{aligned}&\sum_{k=0}^\infty \left\{\frac{1}{z+a+b+2k+1}+\frac{1}{z-a-b+2k+1}-\frac{1}{z+a-b+2k+1}-\frac{1}{z-a+b+2k+1}\right\}\\ &= \sum_{k=0}^\infty \frac{8ab(z+2k+1)}{\{(z+2k+1)^2-a^2-b^2\}^2-4a^2b^2}\\ &= \frac{2ab}{1(z^2-1)+b^2-a^2+}\,\frac{2(1^2-b^2)}{1}+\frac{2(1^2-a^2)}{3(z^2-1)+b^2-a^2+}\,\frac{4(2^2-b^2)}{1}+\frac{4(2^2-a^2)}{5(z^2-1)+b^2-a^2}+\cdots,\end{aligned} \tag{3.7.34}$$

([Bern89], p 158). $D_c := \{(a,b,z) \in \mathbb{C}^3;\ |\arg(z^2-1)| < \pi\} = \{(a,b,z) \in \mathbb{C}^3;\ \mathrm{Re}\,z \neq 0 \text{ and } z \notin (-1,1)\}$, $D_f := \{(a,b,z) \in \mathbb{C}^3;\ \mathrm{Re}\,z > 0 \text{ and } z \notin (0,1)\}$. Dividing by $2a$ and letting $a \to 0$ in (3.7.34) leads to

$$\sum_{k=0}^{\infty}\left\{\frac{1}{(z-b+2k+1)^2}-\frac{1}{(z+b+2k+1)^2}\right\}=\sum_{k=0}^{\infty}\frac{4b(z+2k+1)}{\{(z+2k+1)^2-b^2\}^2}$$
$$=\frac{b}{1(z^2-1)+b^2}\,\underset{+}{}\,\frac{2(1^2-b^2)}{1}\,\underset{+}{}\,\frac{2\cdot 1^2}{3(z^2-1)+b^2}\,\underset{+}{}\,\frac{4(2^2-b^2)}{1}\,\underset{+}{}\,\frac{4\cdot 2^2}{5(z^2-1)+b^2}\,\underset{+}{}\cdots, \tag{3.7.35}$$

([Bern89], p 158). $D_c := \{(b,z)\in\mathbb{C}^2;\ \operatorname{Re} z\neq 0 \text{ and } z\notin(-1,1)\}$, $D_f := \{(b,z)\in\mathbb{C}^2;\ \operatorname{Re} z>0 \text{ and } z\notin(0,1)\}$. The even part of this continued fraction is

$$\sum_{k=0}^{\infty}\left\{\frac{1}{(z-b+2k+1)^2}-\frac{1}{(z+b+2k+1)^2}\right\}=\sum_{k=0}^{\infty}\frac{4b(z+2k+1)}{\{(z+2k+1)^2-b^2\}^2}$$
$$=\frac{b}{1(z^2-b^2+1)}\,\underset{-}{}\,\frac{4(1^2-b^2)1^4}{3(z^2-b^2+5)}\,\underset{-}{}\,\frac{4(2^2-b^2)2^4}{5(z^2-b^2+13)}\,\underset{-}{}\,\frac{4(3^2-b^2)3^4}{7(z^2-b^2+25)}\,\underset{-}{}\cdots. \tag{3.7.36}$$

$D_c := \{(b,z)\in\mathbb{C}^2;\ \operatorname{Re} z\neq 0\}$, $D_f := \{(b,z)\in\mathbb{C}^2;\ \operatorname{Re} z>0\}$. (The numbers 1, 5, 13, 25, … in the denominators have the form $2n^2+2n+1$ for $n=0,1,2,3,\ldots$.)

$$\frac{u-v}{u+v}=\frac{2a^2}{1z}\,\underset{+}{}\,\frac{4a^4+1^4}{3z}\,\underset{+}{}\,\frac{4a^4+2^4}{5z}\,\underset{+}{}\,\frac{4a^4+3^4}{7z}\,\underset{+}{}\cdots \quad \text{where}$$
$$u:=\prod_{k=0}^{\infty}\left\{1+\left(\frac{2a}{z+2k+1}\right)^2\right\},\quad v:=\frac{\Gamma^2\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z+2a+1}{2}\right)\Gamma\left(\frac{z-2a+1}{2}\right)} \tag{3.7.37}$$

([ABJL92], entry 48). $D_c := \{(a,z)\in\mathbb{C};\ \operatorname{Re} z\neq 0\}$, $D_f := \{(a,z)\in\mathbb{C};\ \operatorname{Re} z>0\}$.

$$\frac{u-v}{u+v}=\frac{a^3}{1(2z^2+2z+1)}\,\underset{+}{}\,\frac{a^6-1^6}{3(2z^2+2z+3)}\,\underset{+}{}\,\frac{a^6-2^6}{5(2z^2+2z+7)}\,\underset{+}{}\cdots$$
where
$$u:=\prod_{k=1}^{\infty}\left\{1+\left(\frac{a}{z+k}\right)^3\right\},\quad v:=\prod_{k=1}^{\infty}\left\{1-\left(\frac{a}{z+k}\right)^3\right\}, \tag{3.7.38}$$

([ABJL92], entry 50). $D_c := \{(a,z)\in\mathbb{C}^2;\ \operatorname{Re} z\neq -\frac{1}{2}\}$, $D_f := \{(a,z)\in\mathbb{C}^2;\ \operatorname{Re} z> -\frac{1}{2}\}$. (The numbers $1,3,7,\ldots$ in the denominators are the numbers n^2+n+1 for $n=0,1,2,\ldots$.)

$$2\sum_{k=0}^{\infty}\frac{(-1)^k y^{2k+1}}{r+2k+1}=\frac{z}{1+a}\,\underset{+}{}\,\frac{1^2z^2}{3+a}\,\underset{+}{}\,\frac{2^2z^2}{5+a}\,\underset{+}{}\,\frac{3^2z^2}{7+a}\,\underset{+}{}\cdots \tag{3.7.39}$$
$$\text{where}\quad y:=(\sqrt{1+z^2}-1)/z\quad\text{and}\quad r:=a/\sqrt{1+z^2}$$

for $(a,z)\in\mathbb{C}^2$ with $|\arg(z^2+1)|<\pi$; i.e., $z\in\mathbb{C}\setminus i((-\infty,-1]\cup[1,\infty))$, ([ABJL92], entry 14). With the same notation and same region for (a,z), also

$$y+r\left(y+\frac{1}{y}\right)\sum_{k=1}^{\infty}\frac{(-1)^k y^{2k}}{r+2k}=\frac{z}{2+a}\,\underset{+}{}\,\frac{1\cdot 2z^2}{4+a}\,\underset{+}{}\,\frac{2\cdot 3z^2}{6+a}\,\underset{+}{}\,\frac{3\cdot 4z^2}{8+a}\,\underset{+}{}\cdots, \tag{3.7.40}$$

([ABJL92], entry 15) and

$$\left(1+\frac{1}{z^2}\right)^{(b-1)/2}(2y)^b\sum_{k=0}^{\infty}\frac{(-1)^k(b)_k y^{2k}}{k!(r+b+2k)} = \frac{z}{a+b+}\,\frac{1\cdot bz^2}{a+b+2+}\,\frac{2(b+1)z^2}{a+b+4}\,+\frac{3(b+2)z^2}{a+b+6}+\cdots, \tag{3.7.41}$$

([ABJL92], entry 17), where $b \in \mathbb{C}$.

A.4 Basic hypergeometric functions

In this chapter we use the standard notation

$$_2\varphi_1(a,b;c;q;z) := \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}\,z^n$$

where $(d;q)_0 := 1,\ \ (d;q)_n := (1-d)(1-dq)\cdots(1-dq^{n-1})$ for $n \in \mathbb{N}$.

For convenience we always assume that $q \in \mathbb{C}$ with $|q| < 1$, although the continued fraction may well converge, even to the right value, for other values of $q \in \mathbb{C}$.

A.4.1 General expressions

$$(1-c)\frac{_2\varphi_1(a,b;c;q;z)}{_2\varphi_1(a,bq;cq;q;z)} = 1-c+\frac{(1-a)(c-b)z}{1-cq}\,+\frac{(1-bq)(cq-a)z}{1-cq^2}\,+\frac{(1-aq)(cq-b)qz}{1-cq^3}\,+\frac{(1-bq^2)(cq^2-a)qz}{1-cq^4}\,+\frac{(1-aq^2)(cq^2-b)q^2z}{1-cq^5}\,+\cdots \tag{4.1.1}$$

for $(a,b,c,z) \in \mathbb{C}^4$, ([ABBW85], p 14).

$$(1-c)\frac{_2\varphi_1(a,b;c;q;z)}{_2\varphi_1(aq,bq;cq;q;z)} = b_0+\mathbf{K}(a_n/b_n)$$
$$\text{where } a_n := (1-aq^n)(1-bq^n)cq^{n-1}(1-zabq^n/c)z$$
$$b_n := 1-cq^n-(a+b-abq^n-abq^{n+1})q^n z \tag{4.1.2}$$

for $(a,b,c,z) \in \mathbb{C}^4$.

$$q(1-c)\frac{_2\varphi_1(a,b;c;q;z)}{_2\varphi_1(a,bq;cq;q;z)} = (1-c)q+(a-bq)z-\frac{(a-cq)(1-bq)qz}{(1-cq)q+(a-bq^2)z-}\,\frac{(a-cq^2)(1-bq^2)qz}{(1-cq^2)q+(a-bq^3)z}\,-\frac{(a-cq^3)(1-bq^3)qz}{(1-cq^3)q+(a-bq^4)z}\,-\cdots \tag{4.1.3}$$

for $(a,b,c,z) \in \mathbb{C}^4$, ([ABBW85], p 18).

If we choose $b = 1$ in (4.1.1), (4.1.2) or (4.1.3) we obtain continued fraction expansions for ${}_2\varphi_1(a, q; cq; q; z)$ or ${}_2\varphi_1(aq, q; cq; q; z)$.

A.4.2 Two general results by Andrews

$$\begin{aligned}&\frac{G(a,b,c;q)}{G(aq,b,cq;q)} = 1 + \frac{aq+cq}{1} \underset{+}{} \frac{bq+cq^2}{1} \underset{+}{} \frac{aq^2+cq^3}{1} \underset{+}{} \frac{bq^2+cq^4}{1} \underset{+}{} \cdots \\ &\text{where} \quad G(a,b,c;q) := \sum_{k=0}^{\infty} \frac{(-\frac{c}{a};q)_k q^{k(k+1)/2} a^k}{(q;q)_k(-bq;q)_k}\end{aligned} \tag{4.2.1}$$

for $(a, b, c) \in \mathbb{C}^3$, ([ABJL89], p 80).

$$\begin{aligned}&\frac{H(a_1,a_2;z;q)}{H(a_1,a_2;qz;q)} = 1 + bqz + \frac{(1+aq^2z)qz}{1+bq^2z} \underset{+}{} \frac{(1+aq^3z)q^2z}{1+bq^3z} \underset{+}{} \cdots \\ &\text{where } a := -1/a_1a_2 \text{ and } b := -1/a_1 - 1/a_2 \text{ and} \\ &H(a_1,a_2;z;q) := \frac{\left(\frac{qz}{a_1};q\right)_\infty \left(\frac{qz}{a_2};q\right)_\infty}{(qz;q)_\infty(1-z)} \times \\ &\qquad \times \sum_{k=0}^{\infty} \frac{(1-zq^{2k})(z;q)_k(a_1;q)_k(a_2;q)_k q^{k(3k+1)/2}(az^2)^k}{(q;q)_k \left(\frac{qz}{a_1};q\right)_k \left(\frac{qz}{a_2};q\right)_k}\end{aligned} \tag{4.2.2}$$

for $(1/a_1,\ 1/a_2, z) \in \mathbb{C}^3$, ([ABJL89], p 79).

A.4.3 q-expressions by Ramanujan

The formula (4.2.1) can also be found in Ramanujan's lost notebook ([Andr79], p 90). Quite a number of Ramanujan's expressions are special cases of (4.2.1) and (4.2.2). We refer in particular to ([ABJL92]) for more details. From (4.1.1) we find that

$$\begin{aligned}&\frac{(-a;q)_\infty(b;q)_\infty - (a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty + (a;q)_\infty(-b;q)_\infty} = \frac{a-b}{1-q} \cdot \frac{{}_2\varphi_1\left(\frac{bq}{a}, \frac{bq^2}{a}; q^3; q^2; a^2\right)}{{}_2\varphi_1\left(\frac{bq}{a}, \frac{b}{a}; q; q^2; a^2\right)} \\ &= \frac{a-b}{1-q} \underset{+}{} \frac{(a-bq)(aq-b)}{1-q^3} \underset{+}{} \frac{(a-bq^2)(aq^2-b)q}{1-q^5} \underset{+}{} \frac{(a-bq^3)(aq^3-b)q^2}{1-q^7} \underset{+}{} \cdots\end{aligned} \tag{4.3.1}$$

for $(a, b) \in \mathbb{C}^2$, ([ABBW85], p 14).

$$\frac{(a^2q^3;q^4)_\infty(b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty(b^2q;q^4)_\infty} = \frac{1}{1-ab} \underset{+}{} \frac{(a-bq)(b-aq)}{(1-ab)(q^2+1)} \underset{+}{} \frac{(a-bq^3)(b-aq^3)}{(1-ab)(q^4+1)} \underset{+}{} \cdots \tag{4.3.2}$$

for $(a, b) \in \mathbb{C}^2$, ([ABBW85], entry 12).

$$\begin{aligned} &\frac{F(b;a)}{F(b;aq)} = 1 + \frac{aq}{1+bq+}\frac{aq^2}{1+bq^2+}\frac{aq^3}{1+bq^3+}\cdots \\ &\text{where } F(b;a) := \sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(-bq;q)_k (q;q)_k} \end{aligned} \tag{4.3.3}$$

for $(a, b) \in \mathbb{C}^2$, ([ABBW85], entry 15).

If we set $a := 0$ in (4.2.1) we get

$$\begin{aligned} &\frac{\varphi(c)}{\varphi(cq)} = 1 + \frac{cq}{1+}\frac{bq+cq^2}{1}\,\frac{}{+}\frac{cq^3}{1+}\frac{bq^2+cq^4}{1}\,\frac{}{+}\frac{cq^5}{1+}\cdots \\ &\text{where } \varphi(c) := \sum_{k=0}^{\infty} \frac{q^{k^2} c^k}{(q;q)_k (-bq;q)_k)}, \end{aligned} \tag{4.3.4}$$

([ABJL92], entry 56). If we moreover set $b := -c$, this reduces to

$$\sum_{k=0}^{\infty} (-c)^k q^{k(k+1)/2} = \frac{1}{1+}\frac{cq}{1+}\frac{c(q^2-q)}{1}\,\frac{}{+}\frac{cq^3}{1+}\frac{c(q^4-q^2)}{1}\,\frac{}{+}\cdots, \tag{4.3.5}$$

([ABBW85], p 22).

$$\begin{aligned} &\frac{G(z)}{G(qz)} = 1 - \frac{qz}{1+q+}\frac{q^3z}{1+q^2-}\frac{q^2z}{1+q^3+}\frac{q^6z}{1+q^4-}\frac{q^3z}{1+q^5+}\frac{q^9z}{1+q^6-}\cdots \\ &\text{where } G(z) := \sum_{k=0}^{\infty} \frac{(-z)^k q^{k(k+1)/2}}{(q^2;q^2)_k} \end{aligned} \tag{4.3.6}$$

for $z \in \mathbb{C}$, ([ABJL92], formula 9.1).

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1-}\frac{q}{1+q-}\frac{q^3}{1+q^2-}\frac{q^5}{1+q^3-}\frac{q^7}{1+q^4-}\cdots, \tag{4.3.7}$$

([ABJL92], entry 10).

$$\frac{(q^3;q^4)_\infty}{(q;q^4)_\infty} = \frac{1}{1-}\frac{q}{1+q^2-}\frac{q^3}{1+q^4-}\frac{q^5}{1+q^6-}\cdots, \tag{4.3.8}$$

([ABJL92], entry 11).

$$\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty} = \frac{1}{1+}\frac{q}{1+}\frac{q^2+q}{1}\,\frac{}{+}\frac{q^3}{1+}\frac{q^4+q^2}{1}\,\frac{}{+}\frac{q^5}{1+}\cdots, \tag{4.3.9}$$

([ABJL92], entry 12).

$$\frac{(q;q^2)_\infty}{\{(q^3;q^6)_\infty\}^3} = \frac{1}{1+}\frac{q+q^2}{1}\,\frac{}{+}\frac{q^2+q^4}{1}\,\frac{}{+}\frac{q^3+q^6}{1}\,\frac{}{+}\cdots, \tag{4.3.10}$$

([ABJL89], thm 7).

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\cdots, \tag{4.3.11}$$

([ABJL89], (5)).

$$\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} = \frac{1}{1+}\,\frac{q+q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^3+q^6}{1+}\,\frac{q^8}{1+}\,\frac{q^5+q^{10}}{1+}\cdots, \tag{4.3.12}$$

([ABJL89], thm 6).

$$\sum_{k=1}^{\infty}\frac{(a;q)_\infty a^k}{(q;q)_k(1+q^kz)} = \frac{a}{1+}\,\frac{(1-a)qz}{1+}\,\frac{(1-q)aqz}{1+}\,\frac{(1-aq)q^2z}{1+}\,\frac{(1-q^2)aq^2z}{1+}\,\frac{(1-aq^2)q^3z}{1+}\cdots \tag{4.3.13}$$

for $(a,z)\in\mathbb{C}^2$, ([Wall48], p 376).

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Index